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Oscillation and Nonoscillation Theorems for Certain Second-Order Difference Equations with Forcing Term

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In this paper we are concerned with some new criteria for the oscillation and nonoscillation of the second-order nonhomogeneous linear difference equations of the form $\Delta(c_{n-1}\Delta x_{n-1}) + q_n x_n = f_n$, $n = 1, 2, \ldots,$ where $\{c_n\}$, $\{f_n\}$, and $\{q_n\}$ are real sequences, $c_n > 0$ for $n \geq 0$, and $\Delta x_n = x_{n+1} - x_n$ is the forward difference operator. The discrete analogs of some of the known results in the continuous case are presented.

1. INTRODUCTION

Recently there has been an increasing interest in studying the oscillatory character of difference equations of various types. Among the concerned literature there has been a considerable interest in the oscillatory properties of the second-order linear difference equations of the form

$$\Delta(c_{n-1}\Delta x_{n-1}) + q_n x_n = 0, \qquad n = 1, 2, \ldots \tag{E$_0$}$$

where Δ is the forward difference operator, i.e., $\Delta x_n = x_{n+1} - x_n$, $\{c_n\}$ and $\{q_n\}$ are sequences of real numbers such that $c_n > 0$ for $n \geq 0$. For details, the reader is referred to [1, 2, 4–6, 12] and the references cited therein. However, less is known about the oscillatory properties of the forced equation

$$\Delta(c_{n-1}\Delta x_{n-1}) + q_n x_n = f_n, \qquad n = 1, 2, \ldots \tag{E}$$

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where $\{f_n\}$ is a sequence of real numbers. Therefore, the purpose of the present paper is to investigate the oscillatory behavior of Eq. (E).

By a solution of Eq. (E) we mean an eventually nontrivial sequence $\{u_n\}$ satisfying Eq. (E) for $n = 0, 1, 2, \dots$. As usual, a solution $\{u_n\}$ of Eq. (E) is said to be nonoscillatory if there exists $N \geq 0$ such that $u_n u_{n+1} > 0$ for all $n \geq N$ and is oscillatory otherwise. An equation is called oscillatory if all of its solutions are oscillatory.

It is known that the oscillation of Eq. (E_0) is equivalent to the oscillation of one of its solutions (see [3, p. 153]). However. Eq. (E) can have both oscillatory and nonoscillatory solutions. For example, the equation

$$\Delta^2 u_{n-1} + 4u_n = 1, \qquad n = 1, 2, \dots$$

has a nonoscillatory solution $\{1/4 + (-1)^n/8\}$ and a oscillatory solution $\{1/4 + n(-1)^n\}$. For the general theory of the difference equations, the reader is referred to [3, 7, 10].

2. OSCILLATION AND NONOSCILLATION CRITERIA

In this section, the oscillatory properties of Eq. (E) are studied via the transformation $u_n = x_n y_n$, where $\{x_n\}$ is a solution of Eq. (E_0) and $x_n \neq 0$, $n \geq N$ for some $N \geq 0$. This transformation is the discrete analog of the continuous version $u(t) = x(t)y(t)$, which has been employed by [14] for study of the continuous analog of Eq. (E), i.e., the differential equation

$$(c(t)u'(t))' + q(t)u(t) = f(t), \qquad (' = d/dt),$$

and by [8, 11] for the continuous analog of Eq. (E_0), i.e.,

$$(c(t)u'(t))' + q(t)u(t) = 0.$$

LEMMA 2.1. *Suppose that* $\{u_n\}$ *and* $\{x_n\}$ *are solutions of Eq.* (*E*) *and Eq.* (E_0), *respectively, such that* $x_n \neq 0$, $n \geq N$ *for some* $N \geq 0$. *Define* y_n *by* $u_n = x_n y_n$, $n \geq N$. *Then*

$$\Delta(c_{n-1} x_{n-1} x_n \, \Delta y_{n-1}) = x_n f_n, \qquad n \geq N. \tag{2.1}$$

Proof. Since $u_n = x_n y_n$, then

$$\Delta u_{n-1} = x_{n-1} \, \Delta y_{n-1} + y_n \, \Delta x_{n-1},$$

and

$$\begin{aligned} \Delta(c_{n-1} \, \Delta u_{n-1}) &= \Delta(c_{n-1} x_{n-1} \, \Delta y_{n-1}) + \Delta(c_{n-1} y_n \, \Delta x_{n-1}) \\ &= \Delta(c_{n-1} x_{n-1} \, \Delta y_{n-1}) + y_n \Delta(c_{n-1} \, \Delta x_{n-1}) + c_n \, \Delta y_n \, \Delta x_n. \end{aligned}$$

Using Eq. (E), we obtain

$$\begin{aligned} f_n &= \Delta(c_{n-1}x_{n-1}\,\Delta y_{n-1}) + y_n\Delta(c_{n-1}\,\Delta x_{n-1}) + c_n\,\Delta y_n\,\Delta x_n + q_n x_n y_n \\ &= \Delta(c_{n-1}x_{n-1}\,\Delta y_{n-1}) + c_n\,\Delta y_n\,\Delta x_n + y_n\big[\Delta(c_{n-1}\,\Delta x_{n-1}) + q_n x_n\big], \\ &\qquad n \ge N, \end{aligned}$$

but $\{x_n\}$ is a solution of Eq. (E_0); therefore we have

$$\begin{aligned} f_n &= \Delta(c_{n-1}x_{n-1}\,\Delta y_{n-1}) + c_n\,\Delta y_n\,\Delta x_n \\ &= -c_{n-1}x_{n-1}\,\Delta y_{n-1} + c_n x_{n+1}\,\Delta y_n, \qquad n \ge N. \end{aligned} \tag{2.2}$$

Multiplying both sides of (2.2) by x_n, we get

$$\begin{aligned} x_n f_n &= c_n x_n x_{n+1}\,\Delta y_n - c_{n-1}x_{n-1}x_n\,\Delta y_{n-1} \\ &= \Delta(c_{n-1}x_{n-1}x_n\,\Delta y_{n-1}), \qquad n \ge N. \end{aligned}$$

The proof is complete. ∎

THEOREM 2.1. *If there exists an eventually positive solution $\{x_n\}$ of Eq. (E_0) such that for sufficiently large integer $N \ge 0$ and for some $M > 0$, the conditions*

$$\liminf_{n\to\infty} \sum_{k=N}^{n} x_k f_k = -\infty \quad \text{and} \quad \limsup_{n\to\infty} \sum_{k=N}^{n} x_k f_k = \infty, \tag{2.3}$$

$$\left| \sum_{k=N}^{n} \frac{1}{c_k x_k x_{k+1}} \sum_{i=N}^{k} x_i f_i \right| \le M \sum_{k=N}^{n} \frac{1}{c_k x_k x_{k+1}}, \qquad n \ge N, \tag{2.4}$$

and

$$\sum_{k=N}^{\infty} \frac{1}{c_k x_k x_{k+1}} = \infty, \tag{2.5}$$

are satisfied, then Eq. (E) is oscillatory.

Proof. For the sake of contradiction, we assume that $\{u_n\}$ is a nonoscillatory solution of Eq. (E). Without any loss of generality, one can assume that $\{u_n\}$ is eventually positive, since otherwise the substitution $v_n = -u_n$ transforms Eq. (E) into a similar equation, with f_n replaced by $-f_n$, furthermore, the resulting equation preserves the assumptions of the theorem. Hence we talk again about an eventually positive solution.

Now let N be a sufficiently large integer so that the assumptions of the theorem hold, $u_n > 0$ and $x_n \ne 0$ for $n \ge N$. As in Lemma 2.1, the sequence $\{y_n\}$ defined by $u_n = x_n y_n$ is a solution of (2.1). Summing (2.1) from N to n, we get

$$\sum_{k=N}^{n} x_k f_k = c_n x_n x_{n+1}\,\Delta y_n - c_{N-1}x_{N-1}x_N\,\Delta y_{N-1}. \tag{2.6}$$

By (2.3), we have

$$\liminf_{n\to\infty} c_n x_n x_{n+1} \Delta y_n = -\infty.$$

Thus one can choose N so large that

$$c_{N-1} x_{N-1} x_N \Delta y_{N-1} < -2M. \tag{2.7}$$

Dividing both sides of (2.6) by $c_n x_n x_{n+1}$ and summing from N to n, we obtain

$$\begin{aligned} y_{n+1} &= y_N + c_{N-1} x_{N-1} x_N \Delta y_{N-1} \sum_{k=N}^{n} \frac{1}{c_k x_k x_{k+1}} + \sum_{k=N}^{n} \frac{1}{c_k x_k x_{k+1}} \sum_{i=N}^{k} x_i f_i \\ &< y_N - 2M \sum_{k=N}^{n} \frac{1}{c_k x_k x_{k+1}} + \sum_{k=N}^{n} \frac{1}{c_k x_k x_{k+1}} \sum_{i=N}^{k} x_i f_i, \qquad n \ge N. \end{aligned} \tag{2.8}$$

From (2.4), we obtain

$$y_n < y_N - M \sum_{k=N}^{n} \frac{1}{c_k x_k x_{k+1}}.$$

By (2.5), y_n is eventually negative, which implies that u_n is eventually negative. Then we arrive at a contradiction of the positivity assumption of u_n. This completes the proof. ∎

THEOREM 2.2. *If there exists a positive solution* $\{x_n\}$ *of Eq.* (E_0) *and an integer* $N \ge 0$ *such that*

$$\liminf_{n\to\infty} \sum_{k=N}^{n} \frac{1}{c_k x_k x_{k+1}} \sum_{i=N}^{k} x_i f_i = -\infty, \tag{2.9}$$

$$\limsup_{n\to\infty} \sum_{k=N}^{n} \frac{1}{c_k x_k x_{k+1}} \sum_{i=N}^{k} x_i f_i = \infty, \tag{2.10}$$

and

$$\sum_{k=N}^{\infty} \frac{1}{c_k x_k x_{k+1}} < \infty, \tag{2.11}$$

then Eq. (E) *is oscillatory.*

Proof. Suppose that $\{u_n\}$ is a nonoscillatory solution of Eq. (E). As in the proof of Theorem 2.1, one can assume that $u_n > 0$ for $n \ge N$ and obtain (2.8), i.e.,

$$y_{n+1} = y_N + c_{N-1} x_{N-1} x_N \Delta y_{N-1} \sum_{k=N}^{n} \frac{1}{c_k x_k x_{k+1}} + \sum_{k=N}^{n} \frac{1}{c_k x_k x_{k+1}} \sum_{i=N}^{k} x_i f_i,$$

which, in view of the conditions (2.9) and (2.11), implies that $\lim_{n\to\infty}\inf y_n = -\infty$. Thus we obtain a contradiction of the positivity of u_n. This apparent contradiction completes the proof. ∎

EXAMPLE 2.1. Consider the difference equation

$$\Delta^2 u_{n-1} - u_n = f_n, \qquad n \geq 1. \tag{2.12}$$

The corresponding homogeneous equation, $\Delta^2 x_{n-1} - x_n = 0$, has a positive solution $x_n = a^n$, $a = (3+\sqrt{5})/2$, $n = 0, 1, 2, \dots$. If

$$f_n = (-1)^{n+1}\left[a^{n+1}(2n+1) + a^{n-1}(2n-1)\right],$$

then all of the assumptions of Theorem 2.2 are satisfied, and hence Eq. (2.12) is oscillatory. One such solution is $u_n = (-1)^n a^n$.

Suppose that $\{u_n\}$ and $\{x_n\}$ are any solutions of Eq. (E) and Eq. (E_0), respectively. Define

$$W(x,u)(n) = c_n(x_{n+1}u_n - u_{n+1}x_n).$$

Theorem 6 in [13] states that if $W(x,u)(n)$ is eventually of one sign, then Eq. (E) is oscillatory if and only if Eq. (E_0) is oscillatory. Next, under the transformation $u_n = x_n y_n$, we observe that

$$W(x,u)(n) = -c_n x_n x_{n+1}\,\Delta y_n,$$

and in view of the condition (2.3) or (2.9–2.11), the equality (2.6) or (2.8) implies that $\{c_n x_n x_{n+1}\,\Delta y_n\}$ is oscillatory, i.e., $W(x,u)(n)$ oscillates. Therefore, [13, Th. 6] is not applicable in this case. In fact, Theorem 6 of [13] preserves the oscillatory properties of Eq. (E) and Eq. (E_0), whereas our Theorems 2.1 and 2.2 generate oscillation in Eq. (E).

Next we give a result of the type as Theorem 6 of [13]. We first define a class K of real sequences as follows

$$K = \left\{\{z_n\} : -\infty = \liminf_{n\to\infty} \sum_{k=N}^{n} \frac{1}{c_k z_k z_{k+1}} < \limsup_{n\to\infty} \sum_{k=N}^{n} \frac{1}{c_k z_k z_{k+1}} = \infty,\ z_n \neq 0 \text{ for } n \geq N > 0\right\}$$

THEOREM 2.3. *Suppose that $\{u_n\}$ is any solution of Eq. (E) and $\{x_n\}$ is a solution of Eq. (E_0) such that $x_n \neq 0$, $n \geq N$ for some $N > 0$, and $\{x_n\} \notin K$. Suppose further that*

$$\sum_{k=N}^{\infty} \frac{1}{c_k x_k x_{k+1}} \sum_{i=N}^{k} x_i f_i = \infty. \tag{2.13}$$

Then $\{u_n\}$ is oscillatory if and only if $\{x_n\}$ is oscillatory.

Proof. Suppose that $y_n = u_n/x_n$, $n \geq N$. Then $W(x, u)(n) = -c_n x_n x_{n+1} \Delta y_n$. It will be convenient to assume that $W(x, u)(n)$ is oscillatory, since otherwise we get the conclusion of the theorem using Theorem 6 of [13]. Thus $\{c_n x_n x_{n+1} \Delta y_n\}$ is oscillatory. Consequently, one can choose N so large that the assumptions of the theorem hold and $c_{N-1} x_{N-1} x_N \Delta y_{N-1} \geq 0 (\leq 0)$.

Now since $x_n \neq 0$ for $n \geq N$, then proceeding as in the proof of Theorem 2.1, we get (2.8). On the other hand, the assumption that $\{x_n\} \notin K$ implies that the sum

$$\sum_{k=N}^{n} \frac{1}{c_k x_k x_{k+1}},$$

is either bounded below or bounded above. We consider the first case and the latter goes the same way. Thus there exists a real constant $D \geq 0$ such that

$$\sum_{k=N}^{n} \frac{1}{c_k x_k x_{k+1}} \geq -D, \qquad \text{for all } n \geq N.$$

Choose N so large that $c_{N-1} x_{N-1} x_N \Delta y_{N-1} \geq 0$; then (2.8) implies

$$y_n \geq y_N - D(c_{N-1} x_{N-1} x_N \Delta y_{N-1}) + \sum_{k=N}^{n} \frac{1}{c_k x_k x_{k+1}} \sum_{i=N}^{k} x_i f_i, \qquad n \geq N.$$

In view of (2.13), the above inequality implies that $y_n \to \infty$ as $n \to \infty$. Thus $y_n = u_n/x_n$ is eventually positive, which implies that u_n has the same sign as x_n. This is equivalent to saying that $\{u_n\}$ is oscillatory if and only if $\{x_n\}$ is oscillatory. Thus the proof is complete. ∎

Remark 2.1.

1. If $c_n \equiv 1$, Theorem 2.1 and Theorem 2.2 are the discrete analogs of Theorem 1 and Theorem 2 of [14], respectively.

2. We note that if Eq. (E_0) is nonoscillatory, there exist solutions that satisfy (2.5) and (2.11) (see [12]). Now suppose that Eq. (E_0) is nonoscillatory and $\{x_n\}$ is a positive solution of Eq. (E_0) satisfying (2.5). It is easy to see that (2.13) is satisfied for any eventually positive sequence $\{f_n\}$. Therefore, Theorem 2.3 implies that all of the solutions of Eq. (E) are nonoscillatory, which is the same conclusion as Corollary 4 of [13].

3. EXPLICIT CRITERIA

In this section we give some criteria for the oscillation and nonoscillation of Eq. (E) that depend only on the coefficients $\{c_n\}$, $\{f_n\}$ and/or $\{q_n\}$. We will make use of the following notations: $h_n^+ = \max\{h_n, 0\}$ and $h_n^- =$

$\min\{h_n, 0\}$ for $n = 1, 2, \ldots$. The following result is derived from Theorem 2.3.

THEOREM 3.1. *Suppose that the solutions of Eq.* (E_0) *are bounded and nonoscillatory, and*

$$\sum^{\infty} f_k^+ / c_k = \infty \quad \textit{and} \quad \sum^{\infty} f_k^- > -\infty, \tag{3.1}$$

or

$$\sum^{\infty} f_k^- / c_k = -\infty \quad \textit{and} \quad \sum^{\infty} f_k^+ < \infty. \tag{3.2}$$

Then all of the solutions of Eq. (E) *are nonoscillatory.*

Proof. First, assume that condition (3.1) holds and that $\{x_n\}$ is an eventually positive solution of Eq. (E_0) satisfying (2.11), i.e.,

$$\sum_{k=N}^{\infty} \frac{1}{c_k x_{k+1} x_k} < \infty.$$

Suppose also that the positive constants N, M, and λ are such that $0 < x_n < M$ and $\sum_{k=N}^{n} f_k^- \geq -\lambda$ for all $n \geq N$. Then we get

$$\begin{aligned}
\sum_{k=N}^{n} \frac{1}{c_k x_k x_{k+1}} \sum_{i=N}^{k} x_i f_i &= \sum_{k=N}^{n} \frac{1}{c_k x_k x_{k+1}} \left(\sum_{i=N}^{k} x_i f_i^+ + \sum_{i=N}^{k} x_i f_i^- \right) \\
&\geq \sum_{k=N}^{n} \frac{1}{c_k x_{k+1}} f_k^+ + M \sum_{k=N}^{n} \frac{1}{c_k x_k x_{k+1}} \sum_{i=N}^{k} f_i^- \\
&\geq \frac{1}{M} \sum_{i=N}^{n} \frac{f_k^+}{c_k} - \lambda M \sum_{k=N}^{n} \frac{1}{c_k x_k x_{k+1}}.
\end{aligned}$$

Taking the limit as $n \to \infty$ of both sides of the above inequality and using (3.1), we obtain

$$\sum_{k=N}^{\infty} \frac{1}{c_k x_k x_{k+1}} \sum_{i=N}^{k} x_i f_i = \infty.$$

Thus condition (2.13) is satisfied. Applying Theorem 2.3, we get that any solution of Eq. (E) is nonoscillatory.

Next suppose that condition (3.2) holds. If $\{u_n\}$ is a solution of Eq. (E), it is clear that $\{-u_n\}$ is a solution of the equation

$$\Delta(c_{n-1} \Delta u_{n-1}) + q_n u_n = g_n \qquad n = 1, 2, \ldots, \tag{3.3}$$

where $g_n = -f_n$ for all n. But,

$$\sum^{\infty} g_k^+ / c_k = -\sum^{\infty} f_k^- / c_k = \infty,$$

and

$$\sum_{k=N}^{\infty} g_k^- = -\sum_{k=N}^{\infty} f_k^+ > -\infty,$$

Then in view of the first part of this theorem, any solution of (3.3) is nonoscillatory. Consequently, any solution of Eq. (E) is nonoscillatory. This completes the proof. ∎

Using any known nonoscillation and boundedness criteria of the solutions of Eq. (E_0) and by the use of Theorem 3.1, we can derive many explicit nonoscillation criteria regarding the solutions of Eq. (E). Next, we give a result of this type. We need the following result, which is a simple combination of Theorem 6 in [6] and Theorem 3 in [12]. The result is given by our notation, and the proof is omitted.

THEOREM 3.2. *If for all large n we have*

$$\left.\begin{aligned} &c_n + c_{n-1} > q_n \geq 0 \\ \textit{and} \quad & \\ &\frac{c_n^2}{(c_n + c_{n-1} - q_n)(c_n + c_{n+1} - q_{n+1})} \leq \frac{1}{4}, \end{aligned}\right\} \tag{3.4}$$

and

$$\sum^{\infty} \frac{1}{c_k} < \infty, \tag{3.5}$$

then all solutions of Eq. (E_0) are bounded and nonoscillatory.

Now, by Theorems 3.1 and 3.2, we obtain the following result.

COROLLARY 3.1. *Suppose that conditions* (3.1) (*or* (3.2)), (3.4), *and* (3.5) *are satisfied. Then all the solutions of Eq.* (E) *are nonoscillatory.*

The following example is illustrative.

EXAMPLE 3.1. Consider the following equation:

$$\Delta\left(6^{n-1}\,\Delta u_{n-1}\right) + 26^{n-1}u_n = f_n, \qquad n = 1, 2, \ldots, \tag{3.6}$$

where

$$f_n = n6^n\left(1 + (-1)^n\right)/2 + (2/3)^n\left((-1)^n - 1\right)/2 \qquad \text{for } n = 1, 2, \ldots$$

It is easy to see that conditions (3.4) and (3.5) are satisfied; also,

$$f_n^- = \begin{cases} 0 & \text{if } n \text{ is even} \\ -(2/3)^n & \text{if } n \text{ is odd} \end{cases}$$

and

$$f_n^+ = \begin{cases} n6^n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Then (3.1) and (3.2) are also satisfied. Hence, by Corollary 3.1, all solutions of (3.6) are nonoscillatory. In fact,

$$u_n = (-1)^n(-n/4 + 5/48) - 9/44(-1/9)^n - 9/14(1/9)^n + (3n/2 - 15/4)$$

is one such solution.

In [13] and the preceding results of the present paper, it is observed that the conditions posed for $\{f_n\}$ either preserve the oscillatory properties of Eq. (E_0) in Eq. (E) or yield oscillation in Eq. (E). In either case and for practical purposes, we need first to know not only the oscillatory (nonoscillatory) character of Eq. (E_0), but also some information about the asymptotic behavior of its solutions. So it would be better to find criteria independent of the solutions of Eq. (E_0). Next we include two results of this type.

THEOREM 3.3. *Let $\{n_k\}$ be a sequence of positive integers such that $n_k \to \infty$ as $k \to \infty$. If*

$$c_{n_k} + c_{n_k - 1} - q_{n_k} \le 0 \qquad \text{for all large } k, \tag{3.7}$$

and

$$\{f_{n_k}\} \text{ is oscillatory}, \tag{3.8}$$

then all solutions of Eq. (E) are oscillatory.

Proof. Suppose, for the sake of contradiction, that $\{u_n\}$ is a nonoscillatory solution of Eq. (E). As stated in the proof of Theorem 2.1, we can assume that $\{u_n\}$ is eventually positive. Hence one can choose a sufficiently large integer N such that $u_n > 0$ for $n \ge N$, and (3.7) is satisfied for all $n_k \ge N$. As in [13], Eq. (E) can be written in the following equivalent form:

$$c_n u_{n+1} + c_{n-1} u_{n-1} = (c_n + c_{n-1} - q_n) u_n + f_n, \qquad n \ge N.$$

Hence

$$0 < c_{n_k} u_{n_k+1} + c_{n_k-1} u_{n_k-1} = (c_{n_k} + c_{n_k-1} - q_{n_k}) u_{n_k} + f_{n_k}, \qquad n_k \ge N.$$

Using (3.7),

$$0 < c_{n_k} u_{n_k+1} + c_{n_k-1} u_{n_k-1} \le f_{n_k} \qquad \text{for all } n_k \ge N.$$

Thus, $f_{n_k} > 0$ for all $n_k \ge N$, which contradicts (3.8). This completes the proof. ∎

EXAMPLE 3.2. Consider the equation

$$\Delta^2 u_{n-1} + 2u_n = 2(-1)^n + 1, \qquad n = 1, 2, \ldots$$

Since $c_n + c_{n+1} - q_n = 0$ for all n and $\{f_n\}$ is oscillatory. Then by Theorem 3.3, all solutions of the above equation oscillate. One of these solutions is $u_n = 1/2 - (-1)^n$.

Before stating the last result in this paper, it will be convenient to define the following notations for all $n \ge N > 0$:

$$C_n = \sum_{k=N}^{n} c_k^{-1}, \qquad F_n = C_n^{-1} \sum_{k=N}^{n} c_k^{-1} \sum_{i=N}^{k} f_i.$$

THEOREM 3.4. *Suppose that*

$$q_n \ge 0 \qquad \textit{eventually}, \tag{3.9}$$

$$\lim_{n\to\infty} \sup F_n = \infty, \tag{3.10}$$

and

$$\lim_{n\to\infty} \inf F_n = -\infty, \tag{3.11}$$

Then Eq. (E) is oscillatory.

Proof. Suppose that Eq. (E) is nonoscillatory; then it has solutions that are either eventually positive or eventually negative. Assume that $\{u_n\}$ is one such solution; as in the proof of Theorem 2.1, one can assume that $\{u_n\}$ is eventually positive. Choose an integer $N > 0$ so large that $u_n > 0$ and $q_n \ge 0$ for all $n \ge N$. Now, summing Eq. (E) from N to n and dividing the result by c_n, we get

$$\Delta u_n - \frac{1}{c_n}(c_{N-1}\,\Delta u_{N-1}) + \frac{1}{c_n}\sum_{i=N}^{n} q_i u_i = \frac{1}{c_n}\sum_{i=N}^{n} f_i. \tag{3.12}$$

Summing (3.12) from N to n, we obtain

$$u_{n+1} - u_N - C_n(c_{N-1}\Delta u_{N-1}) + \sum_{k=N}^{n} c_k^{-1} \sum_{i=N}^{k} q_i u_i = \sum_{k=N}^{n} c_k^{-1} \sum_{i=N}^{k} f_i.$$

Dividing both sides of the above inequality by C_n, we have

$$\frac{u_{n+1}}{C_n} - \frac{u_N}{C_n} - c_{N-1}\Delta u_{N-1} + \frac{1}{C_n}\sum_{k=N}^{n} c_k^{-1} \sum_{i=N}^{k} q_i u_i = F_n, \qquad n \geq N. \tag{3.13}$$

Using the positivity of u_n for $n \geq N$, (3.13) implies that

$$-\frac{u_N}{C_n} - c_{N-1}\Delta u_{N-1} \leq F_n, \qquad n \geq N. \tag{3.14}$$

In view of the fact that $C_n > 0$ for $n \geq N$, (3.14) implies that F_n is bounded below, which contradicts (3.11). This completes the proof. ∎

EXAMPLE 3.3. All solutions of the equation

$$\Delta^2 u_{n-1} + n^2 u_n = (-1)^n[4 - n^2], \qquad n = 1, 2, \dots \tag{3.15}$$

are oscillatory by Theorem 3.4; one such solution is $u_n = (-1)^n$.

Remark 3.1.

1. In Theorem 3.3, if we let $f_n \equiv 0$, then Lemma 3 in [12] and Theorem 3.3 are the same. We also observe that the results of this paper (except for Theorem 3.3) are not applicable to Eq. (E_0).

2. It is interesting to obtain a criterion similar to Theorem 3.4 when $q_{n_k} < 0$ for all k, where n_k is defined as in Theorem 3.3.

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