Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2014,**34B**(2):367–379
Mathematica Acta Scientia 数学物理学报
http://actams.wipm.ac.cn

SUBORDINATION PROPERTIES OF MULTIVALENT FUNCTIONS INVOLVING AN EXTENDED FRACTIONAL DIFFERINTEGRAL OPERATOR*

Mohamed K. AOUF †
Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt
E-mail: *mkaouf127@yahoo.com*

Teodor BULBOACĂ
Faculty of Mathematics and Computer Science, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania
E-mail: *bulboaca@math.ubbcluj.ro*

Rabha M. EL-ASHWASH
Department of Mathematics, Faculty of Science, Mansoura University, Mansoura, 35516, Egypt
E-mail: *r_elashwah@yahoo.com*

Abstract The object of this article is to investigate inclusion, radius, and other various properties of subclasses of multivalent analytic functions, which are defined by using an extended version of the Owa-Srivastava fractional differintegral operator $\Omega^{(\lambda,p)}$.

Key words Analytic functions; p–valent functions; meromorphic functions; differintegral operator; differential subordination

2010 MR Subject Classification 26A33; 33C15

1 Introduction

Let $A(p)$ denote the class of functions of the form

$$f(z)=z^p+\sum_{k=1}^{\infty}a_{k+p}z^{k+p},\ p\in\mathbb{N}=\{1,2,\cdots\},\tag{1.1}$$

which are analytic and p–valent in the open unit disc $\mathrm{U}=\{z\in\mathbb{C}:|z|<1\}$. A function $f\in A(p)$ is said to be in the class $S_p^*(\alpha)$ of p–valent starlike functions of order α in U, if

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)}>\alpha,\ z\in\mathrm{U},\quad(0\le\alpha<p).\tag{1.2}$$

Furthermore, a function $f\in A(p)$ is said to be in the class $\mathcal{K}_p(\alpha)$ of p–valent convex functions of order α in U, if

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha,\ z\in\mathrm{U},\quad(0\le\alpha<p).\tag{1.3}$$

*Received August 20, 2012; revised July 10, 2013.
†Corresponding author

From (1.2) and (1.3), it follows that

$$f \in \mathcal{K}_p(\alpha) \Leftrightarrow \frac{zf'(z)}{p} \in S_p^*(\alpha), \quad (0 \le \alpha < p).$$

The classes $S_p^*(\alpha)$ and $\mathcal{K}_p(\alpha)$ were introduced by Kapoor and Mishra [1] (see also [2]). We note that

$$S_p^*(\alpha) \subseteq S_p^*(0) \equiv S_p^* \quad \text{and} \quad \mathcal{K}_p(\alpha) \subseteq \mathcal{K}_p(0) \equiv \mathcal{K}_p,$$

where S_p^* and $\mathcal{K}_p$ denote the subclasses of $A(p)$ consisting of functions which are p–valent starlike in U and p–valent convex in U, respectively (see for details [3], see also [2]).

If f and g are analytic functions in U, we say that f is subordinate to g, written $f(z) \prec g(z)$, if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in$ U, such that $f(z) = g(w(z))$, $z \in$ U. Furthermore, if the function g is univalent in U, then we have the following equivalence (for example, [4], see also [5, p.4]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathrm{U}) \subset g(\mathrm{U}).$$

For the functions $f_i \in A(p)$ $(i = 1, 2)$ given by

$$f_i(z) = z^p + \sum_{k=1}^{\infty} a_{k+p,i}\, z^{k+p},\ z \in \mathrm{U},$$

the Hadamard product (convolution) of f_1 and f_2 is defined by

$$(f_1 * f_2)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p,1}\, a_{k+p,2}\, z^{k+p},\ z \in \mathrm{U}.$$

In this article, we shall also make use of the Gaussian hypergeometric function ${}_2F_1$ defined by

$${}_2F_1(a, b, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad \left(a, b, c \in \mathbb{C},\ c \notin \mathbb{Z}_0^- = \{0, -1, -2, \cdots\}\right), \tag{1.4}$$

where $(d)_k$ denotes the Pochhammer symbol, given in terms of the Gamma function by

$$(d)_k = \frac{\Gamma(d + k)}{\Gamma(d)} = \begin{cases} 1, & \text{if } k = 0,\ d \in \mathbb{C} \setminus \{0\}, \\ d(d + 1) \cdots (d + k - 1), & \text{if } k \in \mathbb{N},\ d \in \mathbb{C}. \end{cases}$$

The series defined by (1.4) converges absolutely for $z \in$ U, and hence ${}_2F_1$ represents an analytic function in U (see, for details, [6, Ch.15]).

Let the integral operator $\mathrm{F}_{\mu,p} : A(p) \to A(p)$, with $\mu > -p$, be defined by

$$\begin{aligned} \mathrm{F}_{\mu,p}(f)(z) &= \frac{\mu + p}{z^\mu} \int_0^z t^{\mu-1} f(t)\, \mathrm{d}\, t = \left(z^p + \sum_{k=1}^{\infty} \frac{\mu + p}{\mu + p + k} z^{p+k} \right) * f(z) \\ &= z^p\ {}_2F_1(1, \mu + p, \mu + p + 1; z) * f(z),\ z \in \mathrm{U}. \end{aligned} \tag{1.5}$$

From definition (1.5), it follows that

$$z\left(\Omega^{(\lambda,p)}\, \mathrm{F}_{\mu,p}(f)(z)\right)' = (p + \mu)\Omega^{(\lambda,p)} f(z) - \mu\Omega^{(\lambda,p)}\, \mathrm{F}_{\mu,p}(f)(z). \tag{1.6}$$

With a view to introducing an extended fractional differintegral operator, we begin by recalling the following definitions of fractional calculus (fractional integral and fractional derivative of an arbitrary order) considered by Owa [7] (see also [8, 9]).

Definition 1.1 The fractional integral of order λ, with $\lambda > 0$, is defined for a function f, analytic in a simply-connected region of the complex plane containing the origin, by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} \,\mathrm{d}\, t,$$

where the multiplicity of $(z-t)^{\lambda-1}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$.

Definition 1.2 Under the hypothesis of Definition 1.1, the fractional derivative of the function f of order λ, with $\lambda \geq 0$, is defined by

$$D_z^{\lambda} f(z) = \begin{cases} \dfrac{1}{\Gamma(1-\lambda)} \dfrac{\mathrm{d}}{\mathrm{d}\, z} \displaystyle\int_0^z \frac{f(t)}{(z-t)^{\lambda}} \,\mathrm{d}\, t, & \text{if } 0 \leq \lambda < 1, \\ \dfrac{\mathrm{d}^n}{\mathrm{d}\, z^n} D_z^{\lambda-n} f(z), & \text{if } n \leq \lambda < n+1,\ n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where the multiplicity of $(z-t)^{-\lambda}$ is removed as in Definition 1.1.

In [10], Patel and Mishra defined the extended fractional differintegral operator $\Omega^{(\lambda,p)} : A(p) \to A(p)$, for a function f of form (1.1) and a real number λ $(\lambda < p+1)$, by

$$\begin{aligned} \Omega^{(\lambda,p)} f(z) &= z^p + \sum_{k=1}^{\infty} \frac{\Gamma(k+p+1)\,\Gamma(p+1-\lambda)}{\Gamma(p+1)\,\Gamma(k+p+1-\lambda)}\, a_{k+p}\, z^{k+p} \\ &= z^p\, {}_2F_1(1, p+1, p+1-\lambda; z) * f(z),\ z \in \mathrm{U}. \end{aligned} \tag{1.7}$$

It is seen from (1.7) that [10]

$$z\left(\Omega^{(\lambda,p)} f(z)\right)' = (p-\lambda)\Omega^{(\lambda+1,p)} f(z) + \lambda \Omega^{(\lambda,p)} f(z),\ z \in \mathrm{U}. \tag{1.8}$$

We also note that

$$\Omega^{(0,p)} f(z) = f(z), \quad \Omega^{(1,p)} f(z) = \frac{zf'(z)}{p},$$

and, in general,

$$\Omega^{(\lambda,p)} f(z) = \frac{\Gamma(p+1-\lambda)}{\Gamma(p+1)} z^{\lambda} D_z^{\lambda} f(z),$$

where $D_z^{\lambda} f$ is, respectively, the fractional integral of f of order $-\lambda$, for $\lambda < 0$, and the fractional derivative of f of order λ, for $0 \leq \lambda < p+1$.

For integer values of λ, relation (1.7) becomes

$$\Omega^{(j,p)} f(z) = \frac{(p-j)! z^j f^{(j)}(z)}{p!}, \quad (j \in \mathbb{N},\ j < p+1),$$

and

$$\begin{aligned} \Omega^{(-m,p)} f(z) &= \frac{p+m}{z^m} \int_0^z t^{m-1} \Omega^{(-m+1,p)} f(t) \,\mathrm{d}\, t = (\mathrm{F}_{1,p} \circ \mathrm{F}_{2,p} \circ \cdots \circ \mathrm{F}_{m,p})(f)(z) \\ &= \mathrm{F}_{1,p}\left(\frac{z^p}{1-z}\right) * \mathrm{F}_{2,p}\left(\frac{z^p}{1-z}\right) * \cdots * \mathrm{F}_{m,p}\left(\frac{z^p}{1-z}\right) * f(z), \quad (m \in \mathbb{N}), \end{aligned}$$

where $\mathrm{F}_{\mu,p}$ is the familiar integral operator defined by (1.5).

The fractional differential operator $\Omega^{(\lambda,p)}$, with $0 \leq \lambda < 1$, was investigated by Srivastava and Aouf [11]. More recently, Srivastava and Mishra [12] obtained several interesting properties and characteristics for certain subclasses of p–valent analytic functions involving the differintegral operator $\Omega^{(\lambda,p)}$, when $\lambda < 1$. Also, the operator $\Omega^{(\lambda,1)} = \Omega^{\lambda}$ was introduced by Owa and Srivastava [8] and the operator Ω^λ is now popularly known as the Owa-Srivastava operator [13–15].

Using the extended fractional differintegral operator $\Omega^{(\lambda,p)}$ with $\lambda < p+1$, we define the following subclass of functions in $A(p)$.

Definition 1.3 For the fixed parameters A and B with $-1 \le B < A \le 1$, $\alpha \ge 0$ and $\lambda < p-1$, with $p > 1$, we say that a function $f \in A(p)$ is in the class $\mathrm{I}^{\lambda}_{p,\alpha}(A,B)$, if

$$(1-\alpha)\frac{\Omega^{(\lambda+1,p)}f(z)}{\Omega^{(\lambda,p)}f(z)} + \alpha\frac{\Omega^{(\lambda+2,p)}f(z)}{\Omega^{(\lambda+1,p)}f(z)} \prec \frac{1+Az}{1+Bz}. \tag{1.9}$$

It is readily seen that

$$\mathrm{I}^0_{p,0}(A,B) = \mathrm{I}^{-1}_{p,1}(A,B) \equiv S^*_p(A,B), \quad \mathrm{I}^{\lambda}_{p}(A,B) \equiv \mathrm{I}^{\lambda}_{p,0}(A,B),$$

and

$$\mathrm{I}^0_{p,1}(A,B) \equiv K_p(A^*,B) = \left\{ f \in A(p) : 1 + \frac{zf''(z)}{f'(z)} \prec p\frac{1+A^*z}{1+Bz} \right\},$$

where

$$A^* = A\left(1-\frac{1}{p}\right) + \frac{B}{p},\ p > 1.$$

2 Preliminaries

To establish our main results, we shall need the following lemmas. The first ones deals with the Briot-Bouquet differential subordinations.

Lemma 2.1 ([16]) Let $\beta, \gamma \in \mathbb{C}$, and let h be a convex function with

$$\mathrm{Re}\,[\beta h(z) + \gamma] > 0,\ z \in \mathrm{U}.$$

If p is analytic in U, with $p(0) = h(0)$, then,

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z).$$

Lemma 2.2 ([17]) Let $\beta > 0$, $\beta + \gamma > 0$ and consider the integral operator $\mathrm{J}_{\beta,\gamma}$ defined by

$$\mathrm{J}_{\beta,\gamma}(f)(z) = \left[\frac{\beta+\gamma}{z^\gamma}\int_0^z f^\beta(t)t^{\gamma-1}\,\mathrm{d}\,t\right]^{\frac{1}{\beta}},$$

where the powers are the principal ones.

If $\sigma \in \left[-\frac{\gamma}{\beta}, 1\right)$, then the order of starlikeness of the class $\mathrm{J}_{\beta,\gamma}\left(S^*(\sigma)\right)$, that is, the largest number $\delta(\sigma;\beta,\gamma)$ such that $\mathrm{J}_{\beta,\gamma}\left(S^*(\sigma)\right) \subset S^*(\delta)$, is given by the number

$$\delta(\sigma;\beta,\gamma) = \inf\{\mathrm{Re}\,q(z) : z \in \mathrm{U}\},$$

where

$$q(z) = \frac{1}{\beta Q(z)} - \frac{\gamma}{\beta} \quad \text{and} \quad Q(z) = \int_0^1 \left(\frac{1-z}{1-tz}\right)^{2\beta(1-\sigma)} t^{\beta+\gamma-1}\,\mathrm{d}\,t.$$

Moreover, if $\sigma \in [\sigma_0, 1)$, where $\sigma_0 = \max\left\{\frac{\beta-\gamma-1}{2\beta}; -\frac{\gamma}{\beta}\right\}$ and $g = \mathrm{J}_{\beta,\gamma}(f)$, with $f \in S^*(\sigma)$, then

$$\mathrm{Re}\,\frac{zg'(z)}{g(z)} > \delta(\sigma;\beta,\gamma),\ z \in \mathrm{U},$$

where

$$\delta(\sigma;\beta,\gamma)=\frac{1}{\beta}\left[\frac{\beta+\gamma}{{}_2F_1\left(1,2\beta(1-\sigma),\beta+\gamma+1;\dfrac{1}{2}\right)}-\gamma\right].$$

Lemma 2.3 ([18]) Let ϕ be analytic in U with $\phi(0)=1$ and $\phi(z)\neq 0$ for $0<|z|<1$, and let $A,B\in\mathbb{C}$ with $A\neq B$, and $|B|\leq 1$.

(i) Let $B\neq 0$ and $\gamma\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}$ satisfy either

$$\left|\frac{\gamma(A-B)}{B}-1\right|\leq 1 \quad \text{or} \quad \left|\frac{\gamma(A-B)}{B}+1\right|\leq 1.$$

If ϕ satisfies

$$1+\frac{z\phi'(z)}{\gamma\phi(z)}\prec\frac{1+Az}{1+Bz},$$

then

$$\phi(z)\prec(1+Bz)^{\frac{\gamma(A-B)}{B}},$$

and this is the best dominant.

(ii) Let $B=0$ and $\gamma\in\mathbb{C}^*$ be such that $|\gamma A|<\pi$, and if ϕ satisfies

$$1+\frac{z\phi'(z)}{\gamma\phi(z)}\prec 1+Az,$$

then

$$\phi(z)\prec e^{\gamma Az},$$

and this is the best dominant.

3 Main Results

Unless otherwise mentioned, we assume throughout this article that $-1\leq B<A\leq 1$, $\alpha\geq 0$, and $\lambda<p-1$, with $p>1$.

Theorem 3.1 Let $-1\leq B<A\leq 1$, $\alpha>0$, let $\lambda<p-1$, $p>1$, and suppose that $f\in A(p)$, with $\Omega^{(\lambda,p)}f(z)\neq 0$ for all $z\in\dot{\mathrm{U}}=\mathrm{U}\setminus\{0\}$.

1. The following implication holds

$$f\in \mathrm{I}^{\lambda}_{p,\alpha}(A,B)\Rightarrow f\in \mathrm{I}^{\lambda}_{p}(\widetilde{A},B),$$

where

$$\widetilde{A}=\frac{(p-\lambda-1)A+\alpha B}{p-\lambda-1+\alpha}. \tag{3.1}$$

2. Moreover, assuming that

$$\frac{1-A}{1-B}\geq\frac{1}{2}-\frac{\alpha}{p-\lambda-1}, \tag{3.2}$$

then

$$f\in \mathrm{I}^{\lambda}_{p,\alpha}(A,B)\Rightarrow f\in \mathrm{I}^{\lambda}_{p}(1-2\rho,-1),$$

where

$$\rho=\rho(A,B)=\frac{1}{{}_2F_1\left(1,\dfrac{2(p-\lambda-1)}{\alpha}\dfrac{A-B}{1-B},\dfrac{p-\lambda-1+2\alpha}{\alpha};\dfrac{1}{2}\right)}, \tag{3.3}$$

and the result is the best possible.

Proof Let $f \in \mathrm{I}_{p,\alpha}^{\lambda}(A,B)$, and put

$$g(z)=\left(\frac{\Omega^{(\lambda,p)}f(z)}{z^p}\right)^{\frac{1}{p-\lambda}}, \tag{3.4}$$

where the power is the principal one. As $\Omega^{(\lambda,p)}f(z)\neq 0$ for all $z\in\dot{\mathrm{U}}$, the function g is analytic in U, with $g(0)=1$. Taking the logarithmic differentiation in (3.4), we have

$$\varphi(z)=\frac{zg'(z)}{g(z)}=\frac{1}{p-\lambda}\left(\frac{z(\Omega^{(\lambda,p)}f(z))'}{\Omega^{(\lambda,p)}f(z)}-p\right),\ z\in\mathrm{U}, \tag{3.5}$$

then, using the identity (1.8) in (3.5), we obtain

$$\frac{\Omega^{(\lambda+1,p)}f(z)}{\Omega^{(\lambda,p)}f(z)}=\varphi(z)+1=\phi(z). \tag{3.6}$$

Logarithmically differentiating both sides of (3.6), and multiplying by z, we deduce that

$$\frac{\Omega^{(\lambda+2,p)}f(z)}{\Omega^{(\lambda+1,p)}f(z)}=\frac{1}{p-\lambda-1}\left[(p-\lambda)\phi(z)-1+\frac{z\phi'(z)}{\phi(z)}\right]. \tag{3.7}$$

Combining (3.7) together with $f\in\mathrm{I}_{p,\alpha}^{\lambda}(A,B)$, we obtain the result that the function ϕ satisfies

$$\left(1+\frac{\alpha}{p-\lambda-1}\right)\phi(z)-\frac{\alpha}{p-\lambda-1}+\frac{\alpha}{p-\lambda-1}\frac{z\phi'(z)}{\phi(z)}\prec\frac{1+Az}{1+Bz}.$$

If we denote

$$P(z)=\left(1+\frac{1}{\beta}\right)\phi(z)-\frac{1}{\beta},\quad \text{where}\quad \beta=\frac{p-\lambda-1}{\alpha},$$

then the above relation is equivalent to the following Briot-Bouquet differential subordination:

$$P(z)+\frac{zP'(z)}{\beta P(z)+1}\prec\frac{1+Az}{1+Bz}\equiv h(z).$$

Now, we will use Lemma 2.1 for the special case $\beta=\dfrac{p-\lambda-1}{\alpha}$ and $\gamma=1$. As the inequality

$$\frac{1-A}{1-B}\geq-\frac{\alpha}{p-\lambda-1} \tag{3.8}$$

holds whenever $-1\leq B<A\leq 1$ and $\lambda<p-1$, using the fact that h is a convex function symmetric with respect to the real axis and (3.8) holds, a simple computation shows that

$$\mathrm{Re}\,[\beta h(z)+\gamma]>\frac{p-\lambda-1}{\alpha}\frac{1-A}{1-B}+1\geq 0,\ z\in\mathrm{U}.$$

Consequently, we have $P(z)\prec h(z)$, that is,

$$\phi(z)\prec\frac{1}{\beta+1}\left[\beta h(z)+1\right]=\frac{1+\widetilde{A}z}{1+Bz},$$

where $\widetilde{A}$ is given by (3.1), or $f\in\mathrm{I}_p^{\lambda}(\widetilde{A},B)$.

If, in addition, we suppose that inequality (3.2) holds, then all the assumptions of Lemma 2.2 are verified for the above values of β,γ, and $\sigma=\frac{1-A}{1-B}$. It follows that $f\in\mathrm{I}_{p,\alpha}^{\lambda}(A,B)$ implies $f\in\mathrm{I}_p^{\lambda}(1-2\rho,-1)$, where the bound $\rho(A,B)$ given by (3.3) is the best possible.

Taking $\alpha=1$, $\lambda=0$ in the second part of Theorem 3.1, if let $A^*=A\left(1-\frac{1}{p}\right)+\frac{B}{p}$, we obtain

Corollary 3.2 Let $-1 \leq B < \frac{pA^*-B}{p-1} \leq 1$ and $p > 1$, such that

$$A^* \leq \frac{p+1+B(p-1)}{2p}.$$

Supposing that $f(z) \neq 0$ for all $z \in \dot{\mathrm{U}}$, then

$$f \in K_p(A^*, B) \Rightarrow f \in S_p^*(p\rho_1),$$

where

$$\rho_1 = \rho_1(A, B) = \frac{1}{{}_2F_1\left(1, \frac{2p(A^*-B)}{1-B}, p+1; \frac{1}{2}\right)},$$

and the result is the best possible.

Remark 3.3 Note that this corollary is a result that relates the order of convexity with the order of starlikeness. It is well known that the class of p–valent convex functions does not have any positive order of starlikeness when $p \geq 2$, and in fact that is why this result is important.

For $\alpha = 1$ and $\lambda = -1$, the second part of Theorem 3.1 reduces to the next result:

Corollary 3.4 Let $-1 \leq B < A \leq 1$, with $p > 1$, such that

$$\frac{1-A}{1-B} \geq \frac{1}{2} - \frac{1}{p}.$$

Supposing that $\int_0^z f(t)\,\mathrm{d}\,t \neq 0$ for all $z \in \dot{\mathrm{U}}$, then,

$$f \in S_p^*(A, B) \Rightarrow \operatorname{Re} \frac{zf(z)}{(p+1)\displaystyle\int_0^z f(t)\,\mathrm{d}\,t} > \rho_2, \ z \in \mathrm{U},$$

where

$$\rho_2 = \rho_2(A, B) = \frac{1}{{}_2F_1\left(1, 2p\frac{A-B}{1-B}, p+2; \frac{1}{2}\right)},$$

and the result is the best possible.

Putting $A = 1 - \frac{2\eta}{p}$, $0 \leq \eta < p$, and $B = -1$ in the second part of Theorem 3.1, we have

Corollary 3.5 Let $\alpha > 0$, $\lambda < p - 1$, with $p > 1$, and let $\eta \in \mathbb{R}$ such that

$$\max\left\{p\left(\frac{1}{2} - \frac{\alpha}{p-\lambda-1}\right); 0\right\} \leq \eta < p. \tag{3.9}$$

Supposing that $\Omega^{(\lambda,p)} f(z) \neq 0$ for all $z \in \dot{\mathrm{U}}$, then,

$$f \in \mathrm{I}_{p,\alpha}^{\lambda}\left(1 - \frac{2\eta}{p}, -1\right) \Rightarrow f \in \mathrm{I}_p^{\lambda}\left(1 - 2\rho_3, -1\right),$$

where

$$\rho_3 = \frac{1}{{}_2F_1\left(1, \frac{2(p-\lambda-1)}{\alpha}\left(1 - \frac{\eta}{p}\right), \frac{p-\lambda-1+2\alpha}{\alpha}; \frac{1}{2}\right)},$$

and the result is the best possible.

Remark 3.6 If we take $\lambda = 0$, $\alpha = \frac{(p-1)\delta}{p-\delta} > 0$, and $\eta = \frac{(\sigma-\delta)p}{p-\delta}$ in Corollary 3.5, it is easy to check that assumption (3.9) holds if and only if $p > 1$, $\delta > 0$, and

$$\max\left\{\frac{p-\delta}{2};\delta\right\} \le \sigma < p.$$

Hence, if $f \in A(p)$, with $f(z) \neq 0$ for all $z \in \dot{\mathrm{U}}$, satisfies

$$\mathrm{Re}\left\{(1-\delta)\frac{zf'(z)}{f(z)} + \delta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \sigma,\ z \in \mathrm{U},$$

then

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} > p\rho_4,\ z \in \mathrm{U}, \quad \text{that is,} \quad f \in S_p^*\left(\rho_4\right),$$

where

$$\rho_4 = \frac{1}{{}_2F_1\left(1, \frac{2(p-\sigma)}{\delta}, \frac{p+\delta}{\delta}; \frac{1}{2}\right)},$$

and the result is the best possible.

We remark that a similar result was also obtained by Patel in [19], and for the special case $p = 1$, the order of starlikeness if the α–convex functions was determined by Miller et al. in [20].

Theorem 3.7 Let $0 \le \alpha \le \frac{\eta(p-\lambda-1)}{p-\eta}$, with $0 \le \eta < p$, $p > 1$, and $\lambda < p-1$. Then, $f \in \mathrm{I}_p^\lambda\left(1 - \frac{2\eta}{p}, -1\right)$ implies $f \in \mathrm{I}_{p,\alpha}^\lambda\left(1 - 2\rho_5, -1\right)$ in $|z| < R(p, \alpha, \lambda, ; \eta)$, where

$$\rho_5 = \frac{-\alpha(p-\eta) + \eta(p-\lambda-1)}{p(p-\lambda-1)},$$

and

$$R(p,\alpha,\lambda;\eta) = \begin{cases} \dfrac{p-\eta}{p-2\eta} + \dfrac{p\alpha - \sqrt{(p\alpha)^2 + (p-\lambda-1+\alpha)\left[(p-\lambda-1+\alpha)\eta^2 + 2p\alpha(p-\eta)\right]}}{(p-\lambda-1+\alpha)(p-2\eta)}, \\ \hfill \text{if} \quad \eta \neq \dfrac{p}{2}, \\ \dfrac{p-\lambda-1+\alpha}{p-\lambda-1+3\alpha}, \hfill \text{if} \quad \eta = \dfrac{p}{2}. \end{cases} \tag{3.10}$$

The result is the best possible.

Proof If $f \in \mathrm{I}_p^\lambda\left(1 - \frac{2\eta}{p}, -1\right)$, and the function u is defined by

$$\frac{\Omega^{(\lambda+1,p)}f(z)}{\Omega^{(\lambda,p)}f(z)} = \frac{\eta}{p} + \left(1 - \frac{\eta}{p}\right)u(z), \tag{3.11}$$

then u is analytic in U, $u(0) = 1$, and has a positive real part in U. Taking the logarithmic derivative of (3.11) and using identity (1.8), after simplification, we have

$$\begin{aligned} &\mathrm{Re}\left[(1-\alpha)\frac{\Omega^{(\lambda+1,p)}f(z)}{\Omega^{(\lambda,p)}f(z)} + \alpha\frac{\Omega^{(\lambda+2,p)}f(z)}{\Omega^{(\lambda+1,p)}f(z)}\right] - \frac{\alpha(\eta-p) + \eta(p-\lambda-1)}{p(p-\lambda-1)} \\ \ge\ &\frac{(p-\eta)(p-\lambda-1+\alpha)}{p(p-\lambda-1)}\left[\mathrm{Re}\,u(z) - \frac{\alpha p\,|zu'(z)|}{(p-\lambda-1+\alpha)\,|\eta + (p-\eta)u(z)|}\right]. \end{aligned}$$

Using in the right-hand side of the above inequality the well-known estimates [21],

$$|zu'(z)| \le \frac{2r}{1-r^2}\,\mathrm{Re}\,u(z) \quad \text{and} \quad \mathrm{Re}\,u(z) \ge \frac{1-r}{1+r},\ \text{for}\ |z| = r < 1,$$

together with the fact that $\mathrm{Re}\,[(p-\eta)u(z)+\eta]>0$ for all $z\in\mathrm{U}$, we obtain

$$\mathrm{Re}\left[(1-\alpha)\frac{\Omega^{(\lambda+1,p)}f(z)}{\Omega^{(\lambda,p)}f(z)}+\alpha\frac{\Omega^{(\lambda+2,p)}f(z)}{\Omega^{(\lambda+1,p)}f(z)}\right]-\frac{\alpha(\eta-p)+\eta(p-\lambda-1)}{p(p-\lambda-1)}$$
$$\geq\frac{(p-\eta)(p-\lambda-1+\alpha)}{p(p-\lambda-1)}\;t(r)\;\mathrm{Re}\,u(z),\ |z|=r<1,$$

where

$$t(r)=1-\frac{2\alpha pr}{(p-\lambda-1+\alpha)\left[\eta(1-r^2)+(p-\eta)(1-r)^2\right]}.$$

A simple computation shows that $t(r)>0$ if $r<R(p,\alpha,\lambda;\eta)$, where $R(p,\alpha,\lambda;\eta)$ is given by (3.10). It is seen that the bound $R(p,\alpha,\lambda;\eta)$ is the best possible, because it is attained for the function $f\in A(p)$ defined by

$$\frac{\Omega^{(\lambda+1,p)}f(z)}{\Omega^{(\lambda,p)}f(z)}=\left(1-\frac{\eta}{p}\right)\frac{1+z}{1-z}+\frac{\eta}{p},\ 0\leq\eta<p.$$

Taking $\alpha=1$ and $\lambda=0$ in Theorem 3.7, we obtain the following result:

Corollary 3.8 If $f\in S_p^*(\eta)$, where $1\leq\eta<p$, then $f\in K_p(\eta)$ for $|z|<R(p,\eta)\equiv R(p,1,0;\eta)$, where

$$R(p,\eta)=\begin{cases}\dfrac{p-\eta+1-\sqrt{\eta^2+2(p-\eta)+1}}{p-2\eta}, & \text{if } \eta\neq\dfrac{p}{2},\\[2ex] \dfrac{p}{p+2}, & \text{if } \eta=\dfrac{p}{2}.\end{cases}$$

The bound $R(p,\eta)$ is the best possible.

Remark 3.9 Note that the result of Corollary 3.8 was previously obtained by Patel and Cho in [22, Corollary 3.3]. This corollary is connected to the classical problem of finding radius of convexity of order η for univalent starlike functions of order η, (see, for example, [23, 24], see also [25]).

Theorem 3.10 Let $-1\leq B<A\leq 1$, $\lambda<p-1$, with $p>1$, and let μ be a real number satisfying $\mu>-p$ and

$$\mu\geq-\lambda-(p-\lambda)\frac{1-A}{1-B}. \tag{3.12}$$

1. Supposing that $\Omega^{(\lambda,p)}\,\mathrm{F}_{\mu,p}(f)(z)\neq 0$ for all $z\in\dot{\mathrm{U}}$, then,

$$f\in\mathrm{I}_p^{\lambda}(A,B)\Rightarrow\mathrm{F}_{\mu,p}(f)\in\mathrm{I}_p^{\lambda}(A,B).$$

2. Moreover, if we suppose in addition that

$$\mu\geq\max\left\{p+2\lambda-1-2(p-\lambda)\frac{1-A}{1-B};-\lambda-(p-\lambda)\frac{1-A}{1-B}\right\}, \tag{3.13}$$

then,

$$f\in\mathrm{I}_p^{\lambda}(A,B)\Rightarrow\mathrm{F}_{\mu,p}(f)\in\mathrm{I}_p^{\lambda}\left(1-2\rho_5,-1\right),$$

where

$$\rho_5=\rho_5(A,B)=\frac{1}{p-\lambda}\left[\frac{p+\mu}{{}_2F_1\left(1,2(p-\lambda)\dfrac{A-B}{1-B},p+\mu+1;\dfrac{1}{2}\right)}-\mu-\lambda\right], \tag{3.14}$$

and the result is the best possible.

Proof If let

$$g(z)=z\left(\frac{\Omega^{(\lambda,p)}\,\mathrm{F}_{\mu,p}(f)(z)}{z^p}\right)^{\frac{1}{p-\lambda}}, \tag{3.15}$$

where the power is the principal one, as $\Omega^{(\lambda,p)}f(z)\neq 0$ for all $z\in\dot{\mathrm{U}}$, the function g is analytic in U, and $g(0)=1$. By carrying out the logarithmic differentiation in (3.15) and using the identity (1.8) for the function $\mathrm{F}_{\mu,p}(f)$, it follows that

$$\varphi(z)=\frac{zg'(z)}{g(z)}=\frac{\Omega^{(\lambda+1,p)}\,\mathrm{F}_{\mu,p}(f)(z)}{\Omega^{(\lambda,p)}\,\mathrm{F}_{\mu,p}(f)(z)} \tag{3.16}$$

is analytic in U and $\varphi(0)=1$. Now, relations (1.8) and (1.6) easily lead to

$$(p-\lambda)\frac{\Omega^{(\lambda+1,p)}\,\mathrm{F}_{\mu,p}(f)(z)}{\Omega^{(\lambda,p)}\,\mathrm{F}_{\mu,p}(f)(z)}+\mu+\lambda=(\mu+p)\frac{\Omega^{(\lambda,p)}f(z)}{\Omega^{(\lambda,p)}\,\mathrm{F}_{\mu,p}(f)(z)}, \tag{3.17}$$

so (3.16) and (3.17) give

$$\frac{\Omega^{(\lambda,p)}\,\mathrm{F}_{\mu,p}(f)(z)}{\Omega^{(\lambda,p)}f(z)}=\frac{p+\mu}{(p-\lambda)\varphi(z)+\mu+\lambda}.$$

Taking the logarithmic differentiation in the above expression and using (1.6) in the resulting equation, we get

$$\frac{\Omega^{(\lambda+1,p)}f(z)}{\Omega^{(\lambda,p)}f(z)}=\varphi(z)+\frac{z\varphi'(z)}{(p-\lambda)\varphi(z)+\mu+\lambda}. \tag{3.18}$$

Hence, by the assumptions of the theorem and (3.18), we have

$$\varphi(z)+\frac{z\varphi'(z)}{\beta\varphi(z)+\gamma}\prec\frac{1+Az}{1+Bz}\equiv h(z), \tag{3.19}$$

where $\beta=p-\lambda$ and $\gamma=\mu+\lambda$. As h is a convex function in U, a simple computation shows that

$$\mathrm{Re}\,[\beta h(z)+\gamma]>(p-\lambda)\frac{1-A}{1-B}+\mu+\lambda\geq 0,\ z\in\mathrm{U},$$

whenever (3.12) holds. Then, from Lemma 2.1 it follows $\varphi(z)\prec h(z)$, that is, $\mathrm{F}_{\mu,p}(f)\in\mathrm{I}_p^{\lambda}(A,B)$.

Supposing in addition that inequality (3.13) holds, then all the assumptions of Lemma 2.2 are satisfied for these values of β,γ, and $\sigma=\frac{1-A}{1-B}$. It follows that $f\in\mathrm{I}_p^{\lambda}(A,B)$ implies $\mathrm{F}_{\mu,p}(f)\in\mathrm{I}_p^{\lambda}(1-2\rho_5,-1)$, where the bound $\rho_5(A,B)$ given by (3.14) is the best possible.

Taking $\lambda=0$ in Theorem 3.10, we obtain the next special case:

Corollary 3.11 Let $-1\leq B<A\leq 1$, $p>1$, and let $\mu\in\mathbb{R}$ satisfying

$$\mu\geq-\frac{p(1-A)}{1-B}.$$

1. Supposing that $\mathrm{F}_{\mu,p}(f)(z)\neq 0$ for all $z\in\dot{\mathrm{U}}$, then,

$$f\in S_p^*(A,B)\Rightarrow\mathrm{F}_{\mu,p}(f)\in S_p^*(A,B).$$

2. Moreover, if we suppose in addition that

$$\mu\geq\max\left\{p-1-\frac{2p(1-A)}{1-B};-\frac{p(1-A)}{1-B}\right\},$$

then,

$$f\in S_p^*(A,B)\Rightarrow\mathrm{F}_{\mu,p}(f)\in S_p^*\left(1-2\rho_6,-1\right),$$

where

$$\rho_6 = \rho_6(A, B) = \frac{1}{p}\left[\frac{p+\mu}{{}_2F_1\left(1, \dfrac{2p(A-B)}{1-B}, p+\mu+1; \dfrac{1}{2}\right)} - \mu\right],$$

and the result is the best possible.

Remark 3.12 We note that Corollary 3.11 improves the result obtained by Patel in [19, Corollary 4].

For $A = 1 - \frac{2\eta}{p}$, $0 \le \eta < p$, and $B = -1$, Corollary 3.11 reduces to

Corollary 3.13 Let $0 \le \eta < p$, $p > 1$, and let $\mu \in \mathbb{R}$ satisfying $\mu \ge -\eta$.

1. Supposing that $\mathrm{F}_{\mu,p}(f)(z) \ne 0$ for all $z \in \dot{\mathrm{U}}$, then,

$$f \in S_p^*(\eta) \Rightarrow \mathrm{F}_{\mu,p}(f) \in S_p^*(\eta).$$

2. Moreover, if we suppose in addition that $\mu \ge \max\{p-1-2\eta; -\eta\}$, then,

$$f \in S_p^*(\eta) \Rightarrow \mathrm{F}_{\mu,p}(f) \in S_p^*(p\rho_7),$$

where

$$\rho_7 = \rho_7(\eta) = \frac{1}{p}\left[\frac{p+\mu}{{}_2F_1\left(1, 2(p-\eta), p+\mu+1; \dfrac{1}{2}\right)} - \mu\right],$$

and the result is the best possible.

Remark 3.14 Note that Corollary 3.13 improves the result obtained by Patel et al. [26, Corollary4].

Theorem 3.15 Let $\lambda < p$, $p \in \mathbb{N}$, let $\nu \in \mathbb{C}^*$ and $A, B \in \mathbb{C}$, with $A \ne B$ and $|B| \le 1$. Suppose that

$$\begin{array}{ll} \left|\dfrac{\nu(p-\lambda)(A-B)}{B} - 1\right| \le 1 \quad \text{or} \quad \left|\dfrac{\nu(p-\lambda)(A-B)}{B} + 1\right| \le 1, & \text{if } B \ne 0, \\ |\nu A| < \dfrac{\pi}{p-\lambda}, & \text{if } B = 0. \end{array}$$

If $f \in A(p)$ with $\Omega^{(\lambda,p)} f(z) \ne 0$ for all $z \in \dot{\mathrm{U}}$, then,

$$\frac{\Omega^{(\lambda+1,p)} f(z)}{\Omega^{(\lambda,p)} f(z)} \prec \frac{1+Az}{1+Bz} \Rightarrow \left(\frac{\Omega^{(\lambda,p)} f(z)}{z^p}\right)^{\nu} \prec q_1(z),$$

where

$$q_1(z) = \begin{cases} (1+Bz)^{\frac{\nu(p-\lambda)(A-B)}{B}}, & \text{if } B \ne 0, \\ e^{\nu(p-\lambda)Az}, & \text{if } B = 0 \end{cases}$$

is the best dominant (The powers are the principal ones).

Proof If let

$$\varphi(z) = \left(\frac{\Omega^{(\lambda+1,p)} f(z)}{z^p}\right)^{\nu}, \tag{3.20}$$

where the power is the principal one, then φ is analytic in U, $\varphi(0) = 1$, and $\varphi(z) \ne 0$ for $z \in \mathrm{U}$. Taking the logarithmic derivatives in both sides of (3.20), multiplying by z and using identity

(1.8), we have

$$1+\frac{z\varphi'(z)}{\nu(p-\lambda)\varphi(z)}=\frac{\Omega^{(\lambda+1,p)}f(z)}{\Omega^{(\lambda,p)}f(z)}\prec\frac{1+Az}{1+Bz}.$$

Now, the assertions of the theorem follows by using Lemma 2.2 for $\gamma=\nu(p-\lambda)$, which completes the proof.

Putting $A=1-2\rho$, $0\le\rho<1$, and $B=-1$ in Theorem 3.15, we obtain the following corollary:

Corollary 3.16 Assume that $\lambda<p$, $p\in\mathbb{N}$, $0\le\rho<1$, and $\nu\in\mathbb{C}^*$ satisfies either

$$|2\nu(p-\lambda)(1-\rho)-1|\le 1 \quad \text{or} \quad |2\nu(p-\lambda)(1-\rho)+1|\le 1.$$

If $f\in A(p)$ with $\Omega^{(\lambda,p)}f(z)\neq 0$ for all $z\in\dot{\mathrm{U}}$, then,

$$\mathrm{Re}\,\frac{\Omega^{(\lambda+1,p)}f(z)}{\Omega^{(\lambda,p)}f(z)}>\rho,\ z\in\mathrm{U}\Rightarrow\left(\frac{\Omega^{(\lambda,p)}f(z)}{z^p}\right)^{\nu}\prec q_2(z),$$

where $q_2=(1-z)^{-2\nu(p-\lambda)(1-\rho)}$ is the best dominant (The powers are the principal ones).

For $\lambda=0$, $A=1-\frac{2\eta}{p}$, $0\le\eta<p$, and $B=-1$, Theorem 3.15 reduces to

Corollary 3.17 Assume that $\nu\in\mathbb{C}^*$ and $0\le\eta<p$, $p\in\mathbb{N}$, satisfies either

$$|2\nu(\eta-p)-1|\le 1 \quad \text{or} \quad |2\nu(\eta-p)+1|\le 1.$$

If $f\in A(p)$ with $f(z)\neq 0$ for all $z\in\dot{\mathrm{U}}$, then,

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)}>\eta,\ z\in\mathrm{U}\Rightarrow\left(\frac{f(z)}{z^p}\right)^{\nu}\prec q_3(z)=(1-z)^{-2\nu(\eta-p)},$$

and q_3 is the best dominant (The powers are the principal ones).

Remark 3.18 Putting $p=1$ in Corollary 3.17, we obtain the corresponding result of Obradović et al. [27, Theorem 1], with $b=1-\eta$, $0\le\eta<1$.

References

[1] Kapoor G P, Mishra A K. Convex hulls and extreme points of some classes of multivalent functions. J Math Anal Appl, 1982, **87**(1): 85–92

[2] Srivastava H M, Owa S. Current Topics in Analytic Function Theory. Singapore, New Jersey, London and Hong Kong: World Scientific Publishing Company, 1992

[3] Goodman A W. On the Schwarz-Christoffel transformation and p–valent functions. Trans Amer Math Soc, 1950, **68**: 204–223

[4] Miller S S, Mocanu P T. Second order differential inequalities in the complex plane. J Math Anal Appl, 1978, **65**: 289–305

[5] Miller S S, Mocanu P. T. Differential Subordinations: Theory and Applications. Series on Monographs and Textbooks in Pure and Applied Mathematics. Vol. 225. New York and Basel: Marcel Dekker, 2000

[6] Whittaker E T, Watson G N. A Course on Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; With an Account of the Principal Transcendental Functions. Fourth Edition. Cambridge: Cambridige Univ Press, 1927

[7] Owa S. On the distortion theorems I. Kyungpook Math J, 1978, **18**: 53–59

[8] Owa S, Srivastava H M. Univalent and starlike generalized hypergeometric functions. Canad J Math, 1987, **39**: 1057–1077

[9] Srivastava H M, Owa S. Univalent Functions, Fractional Calculus, and Their Applications. Halsted Press (Ellis Horwood Limited, Chichester); New York: John Wiley, 1989

[10] Patel J, Mishra A K. On certain subclasses of multivalent functions associated with an extended fractional differintegral operator. J Math Anal Appl, 2007, **332**: 109–122

[11] Srivastava H M, Aouf M K. A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients I and II. J Math Anal Appl, 1992, **171**: 1–13; 1995, **19**: 673–688

[12] Srivastava H M, Mishra A K. A fractional differintegral operator and its applications to a nested class of multivalent functions with negative coefficients. Adv Stud Contemp Math (Kyungshang), 2003, **7**: 203–214

[13] Mishra A K, Gochhayat P. Applications of Owa-Srivastava operator to the class of k–uniformly convex functions. Fract Calc Appl Anal, 2006, **9**: 323–331

[14] Mishra A K, Gochhayat P. The Fekete-Szegő problem for k–uniformly convex functions and for a class defined by the Owa-Srivastava operator. J Math Anal Appl, 2008, **347**: 563–572

[15] Srivastava H M, Mishra A K, Kund S N. Certain classes of analytic functions associated with iterations of the Owa-Srivastava fractional derivative operator. Southeast Asian Bull Math (in press)

[16] Eenigenburg P, Miller S S, Mocanu P T, Reade M O. On a Briot-Bouquet differential subordination//General Inequalities 3, ISNM, Vol. 64, 339–348. Basel: Birkhäuser Verlag, 1983

[17] Mocanu P T, Ripeanu D, Şerb I. The order of starlikeness of certain integral operators. Mathematica (Cluj), 1981, **23**(46): 225–230

[18] Obradović M, Owa S. On certain properties for some classes of starlike functions. J Math Anal Appl, 1990, **145**(2): 357–364

[19] Patel J. On certain subclasses of multivant functions involving Cho-Kwon-Srivastava operator. Ann Univ Mariae Curie-Skłodowska Sect A, 2006, **60**: 75–86

[20] Miller S S, Mocanu P T, Reade M O. The order of starlikeness of α–convex functions. Mathematica (Cluj), 1978, **20**(43): 25–30

[21] MacGregor T H. The radius of univalence of certain analytic functions. Proc Amer Math Soc, 1963, **14**: 514–520

[22] Patel J, Cho N E. Some classes of analytic functions involving Noor integral operator. J Math Anal Appl, 2005, **312**: 564–575

[23] Goel R M. On radii of starlikeness, convexity, close-to-convexity for p–valent functions. Arch Ration Mech Anal, 1972, **44**(4): 320–328

[24] Bajpai S K, Dwivedi S P. On the radii of starlikeness, convexity and close-to-convexity of p–valent functions. Rev Roumaine Math Pures Appl, 1978, **23**(6): 839–842

[25] Goodman A W. Univalent Functions. Vol. 2. Tampa, Florida: Mariner Publishing, 1983

[26] Patel J, Cho N E, Srivastava H M. Certain subclasses of multivalent functions associated with a family of linear operators. Math Comput Modelling, 2006, **43**: 320–338

[27] Obradović M, Aouf M K, Owa S. On some results for starlike functions of complex order. Publ Inst Math Belgrade, 1989, **46**(60): 79–85