Quasi-Hadamard product of some uniformly analytic and $p$-valent functions with negative coefficients

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ABSTRACT. In this paper we study the quasi-Hadamard product between some $p$-valent and uniformly analytic functions with negative coefficients defined in connection with uniformly starlikeness and uniformly convexity.

1. INTRODUCTION

Let $T_0(p)$ denote the class of functions of the form:

$$f(z) = a_p z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad (a_{p+n} \geq 0; p \in \mathbb{N} = \{1, 2, \ldots\}, a_p > 0)$$

which are analytic and $p$-valent in the open unit disc $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. By $T(p)$ we will denote the class $T_0(p)$ with $a_p = 1$.

Also let $S(p,q,\alpha)$ and $C(p,q,\alpha)$ denote two subclasses of $T(p)$ defined as:

$$S(p,q,\alpha) = \left\{ f(z) \in T(p) : \Re \left( \frac{zf^{(1+q)}(z)}{f^{(q)}(z)} \right) > \alpha, \quad (z \in U; 0 \leq \alpha < p - q; p \in \mathbb{N}; p > q; q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}) \right\}$$

and

$$C(p,q,\alpha) = \left\{ f(z) \in T(p) : \Re \left( 1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} \right) > \alpha, \quad (z \in U; 0 \leq \alpha < p - q; p \in \mathbb{N}; p > q; q \in \mathbb{N}_0) \right\}$$

These classes were investigated by Chen et al. in [2]. We can notice that:

(i) $S(p,0,\alpha) = T^*(p,\alpha)$, is the class of $p$-valently starlike functions of order $\alpha$, $0 \leq \alpha < p$;

(ii) $C(p,0,\alpha) = C(p,\alpha)$, is the class of $p$-valently convex functions of order $\alpha$, $0 \leq \alpha < p$.

These last particular classes, $T^*(p,\alpha)$ and $C(p,\alpha)$, were studied by Owa in [7], Sălăgean et al. in [8] and Sekine in [9].

For the classes $S(p,q,\alpha)$ and $C(p,q,\alpha)$, Chen et al. obtained the following results:

Lemma 1.1. [2] A function $f(z) \in T(p)$ is in the class $S(p,q,\alpha)$ if and only if

$$\sum_{n=1}^{\infty} \left( n + p - q - \alpha \right) \delta(n + p, q) a_{p+n} \leq (p - q - \alpha) \delta(p, q)$$
Lemma 1.2. [2] A function \( f(z) \in T(p) \) is in the class \( C(p, q, \alpha) \) if and only if

\[
\sum_{n=1}^{\infty} \left( \frac{n + p - q}{p - q} \right) (n + p - q - \alpha) \delta(n + p, q) a_{p+n} \leq (p - q - \alpha) \delta(p, q).
\]

where

\[
\delta(p, q) = \frac{p!}{(p-q)!} = \left\{ \begin{array}{ll}
p(p-1)\ldots(p-q+1) & (q \neq 0) \\
1 & (q = 0).
\end{array} \right.
\]

Lemma 1.2. \([2]\) A function \( f(z) \in T(p) \) is in the class \( C(p, q, \alpha) \) if and only if

\[
\sum_{n=1}^{\infty} \left( \frac{n + p - q}{p - q} \right) (n + p - q - \alpha) \delta(n + p, q) a_{p+n} \leq (p - q - \alpha) \delta(p, q).
\]

where

\[
\delta(p, q) = \frac{p!}{(p-q)!} = \left\{ \begin{array}{ll}
p(p-1)\ldots(p-q+1) & (q \neq 0) \\
1 & (q = 0).
\end{array} \right.
\]

In order to present the main results of this paper, we introduce two new classes of \( p \)-valent functions with negative coefficients defined by some conditions related to uniformly starlikeness and uniformly convexity, in the following manner:

\[
\beta - UST_0(p, q, \alpha) = \left\{ f \in T_0(p) : \text{Re} \left\{ \frac{zf^{(1+q)}(z)}{f(q)(z)} \right\} \geq \beta \left| \frac{zf^{(1+q)}(z)}{f(q)(z)} - 1 \right| + \alpha \right\},
\]

\( (z \in U; 0 \leq \alpha < p - q; p \in N; p > q; q \in N_0 = N \cup \{0\}, \beta \geq 0) \)

and

\[
\beta - UCV_0(p, q, \alpha) = \left\{ f \in T_0(p) : \text{Re} \left\{ 1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} \right\} \geq \beta \left| \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} \right| + \alpha \right\},
\]

\( (z \in U; 0 \leq \alpha < p - q; p \in N; p > q; q \in N_0 = N \cup \{0\}, \beta \geq 0) \).

Remark 1.1. i) For \( \beta = 0 \) we have \( \beta - UST_0(p, q, \alpha) = S_0(p, q, \alpha) \) and \( \beta - UCV_0(p, q, \alpha) = C_0(p, q, \alpha) \), analogous to \( S(p, q, \alpha) \) and \( C(p, q, \alpha) \), but defined for functions from \( T_0(p) \), the classes considered and investigated by El Ashwah et al. in [3] and for \( \beta = 0 \) and \( a_p = 1 \) we get the classes \( S(p, q, \alpha) \) and \( C(p, q, \alpha) \), studied by Chen et al. in [2].

ii) For \( p = 1, q = 0 \) the classes \( \beta - UST_0(\alpha) \) and \( \beta - UCV_0(\alpha) \) are the classes of \( \beta \)-uniformly starlike functions of order \( \alpha \) and \( \beta \)-uniformly convex functions of order \( \alpha \) introduced by Bharati et al. for \( a_p = 1 \) in [1] and by Frasin for \( a_p \neq 1 \) in [4].

iii) For \( p = 1, q = 0, \alpha = 0 \) we get the classes of \( \beta - UCV \) and \( \beta - UST \) of \( \beta \) uniformly convex and starlike functions introduced by Kanas and Wisniowska in the papers [5] and [6].

iv) For \( p = 1, q = 0, \beta = 0 \) we get the well known classes of convex respectively starlike functions of order \( \alpha \).

v) The new introduced classes are nontrivial generalizations of the above mentioned classes \( \left( \text{consider for example, the function } f(z) = z^2 - \frac{1}{6} z^3 \right) \).
2. Main results

First, we obtain coefficients estimates for the classes $\beta - UST_0(p, q, \alpha)$ and $\beta - UCV_0(p, q, \alpha)$.

**Theorem 2.1.** A function $f \in \beta - UST_0(p, q, \alpha)$ satisfies the inequality

\[
\sum_{n=1}^{\infty} [(n+p-q-\alpha) + \beta(n+p-q-1)] \cdot \delta(n+p,q) \cdot a_{n+p} \leq [(p-q-\alpha) + \beta(p-q-1)] \cdot \delta(p,q) \cdot a_p,
\]

where

\[
\delta(p,q) = \frac{p!}{(p-q)!} = \begin{cases} \frac{p(p-1)\ldots(p-q+1)}{(q\neq 0)} & \\ 1 & (q = 0) \end{cases}.
\]

**Proof.** For the begining we prove (2.1) for $a_p = 1$. Since $f \in \beta - UST_0(p, q, \alpha)$ we get

\[
\Re \left\{ \frac{zf^{(1+q)}(z)}{f^{(q)}(z)} \right\} \geq \beta \left| \frac{zf^{(1+q)}(z)}{f^{(q)}(z)} \right| - 1 \geq \alpha
\]

\( (z \in U; 0 \leq \alpha < p-q; p \in \mathbb{N}; p > q; q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \beta \geq 0 ) \).

Since $|w| \geq -\Re w$ from (2.2) we obtain

\[
\Re \left\{ \frac{zf^{(1+q)}(z)}{f^{(q)}(z)} \right\} \geq \frac{\alpha + \beta}{1 + \beta}
\]

We can notice that $0 \leq \gamma = \frac{\alpha + \beta}{1 + \beta} < p-q$, hence we can say that $f \in S(p,q,\gamma)$.

From Lemma 1.1 it follows that

\[
\sum_{n=1}^{\infty} (n+p-q-\gamma) \delta(n+p,q) a_{p+n} \leq (p-q-\gamma) \delta(p,q)
\]

\( (0 \leq \gamma < p-q; p \in \mathbb{N}; p > q; q \in \mathbb{N}_0) \).

Now if we put $\gamma = \frac{\alpha + \beta}{1 + \beta}$ the inequality (2.3) is equivalent with (2.1) (for $a_p = 1$) so the proof is done for $a_p = 1$.

When $a_p \neq 1$ we have $f(z) = a_p \cdot g(z)$ with $g(z)$ of the form

\[
g(z) = z^p - \sum_{n=1}^{\infty} \frac{a_{p+n}}{a_p} z^{p+n}
\]

satisfying the requested inequality hence $f$ satisfies (2.1) and the proof is complete. \( \square \)

**Theorem 2.2.** A function $f \in \beta - UCV_0(p, q, \alpha)$ satisfies the inequality

\[
\sum_{n=1}^{\infty} \left( \frac{n+p-q}{p-q} \right) [(n+p-q-\alpha) + \beta(n+p-q-1)] \cdot \delta(n+p,q) \cdot a_{n+p}
\]

\[ \leq [(p-q-\alpha) + \beta(p-q-1)] \cdot \delta(p,q) \cdot a_p, \]

where

\[
\delta(p,q) = \frac{p!}{(p-q)!} = \begin{cases} \frac{p(p-1)\ldots(p-q+1)}{(q\neq 0)} & \\ 1 & (q = 0) \end{cases}.
\]
Proof. From the same reasons as in Theorem 2.1 it is sufficient to prove the inequality (2.4) for \( a_p = 1 \). Let \( f \in \beta - UCV_0(p, q, \alpha) \), with \( a_p = 1 \). Then we have

\[
\text{Re} \left\{ 1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} \right\} \geq \beta \left| \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} \right| + \alpha.
\]

Further we derive

\[
\text{Re} \left\{ \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} \right\} \geq \frac{\alpha - 1}{1 + \beta},
\]

and if we denote \( \gamma - 1 = \frac{\alpha - 1}{1 + \beta} \), \( 0 \leq \gamma < p - q \), we have

\[
\text{Re} \left\{ 1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} \right\} \geq \gamma.
\]

Here we can recognize that \( f \in C(p, q, \gamma) \). Then from Lemma 1.2 it follows the inequality (2.5)

\[
\sum_{n=1}^{\infty} \left( \frac{n+p-q}{p-q} \right)^k (n+p-q-\gamma) \delta(n+p,q)a_{p+n} \leq (p-q-\gamma)\delta(p,q).
\]

If we take in (2.5), \( \gamma = \frac{\beta + \alpha}{\beta + 1} \), then we obtain the inequality (2.4), for \( a_p = 1 \) and the proof is complete. \( \square \)

In what follows we introduce a subclass of the class \( T_0(p) \). We say that a function \( f(z) \) from \( T_0(p) \) belongs to the class \( \beta - T_0(k, p, q, \alpha) \) if and only if

\[
\sum_{n=1}^{\infty} \left( \frac{n+p-q}{p-q} \right)^k [(n+p-q-\alpha) + \beta(n+p-q-1)] \delta(n+p,q)a_{p+n} \leq [(p-q-\alpha) + \beta(p-q-1)] \delta(p,q)a_p,
\]

\[
0 \leq \alpha < p-q, \; p \in \mathbb{N}, q \in \mathbb{N} \cup \{0\}, \beta \geq 0, \; \text{where} \; k \; \text{is any fixed non-negative real number.}
\]

Remark 2.2. i) We can notice that the class \( \beta - T_0(k, p, q, \alpha) \) is nonempty since the function of the form

\[
f(z) = a_p z^p - \sum_{n=1}^{\infty} \left( \frac{n+p-q}{p-q} \right)^k \frac{[(p-q-\alpha) + \beta(p-q-1)] \delta(p,q)a_p}{[(n+p-q-\alpha) + \beta(n+p-q-1)] \delta(n+p,q)} \cdot \lambda_{p+n} z^{p+n},
\]

where \( a_p > 0, \lambda_{p+n} > 0 \) and \( \sum_{n=1}^{\infty} \lambda_{p+n} \leq 1 \), satisfies the inequality (2.6).

ii) For \( k > c \geq 0, \) we have that

\[
\beta - T_0(k, p, q, \alpha) \subset \beta - T_0(c, p, q, \alpha).
\]

iii) Also we have the inclusion relations

\[
\beta - UST_0(p, q, \alpha) \subset \beta - T_0(0, p, q, \alpha)
\]

\[
\beta - UCV_0(p, q, \alpha) \subset \beta - T_0(1, p, q, \alpha).
\]
Now, in order to study the quasi-Hadamard product, let the functions $f_i$ and $g_j$ from the class $T_0(p)$ be of the forms:

$$f_i(z) = a_{p,i}z^p - \sum_{n=1}^{\infty} a_{p+n,i}z^{p+n} \quad (a_{p,i} > 0; a_{p+n,i} \geq 0)$$

$$g_j(z) = b_{p,j}z^p - \sum_{n=1}^{\infty} b_{p+n,j}z^{p+n} \quad (b_{p,j} > 0; b_{p+n,j} \geq 0),$$

The quasi-Hadamard product $f_i \ast g_j(z)$ of the functions $f_i(z)$ and $g_j(z)$ is defined by

$$f_i \ast g_j(z) = a_{p,i}b_{p,j}z^p - \sum_{n=1}^{\infty} a_{p+n,i}b_{p+n,j}z^{p+n} \quad (i, j = 1, 2, 3, \ldots).$$

The quasi-Hadamard product (almost a Hadamard product excepting that the minus sign is kept in front of the sum) has been studied before in various papers, among we recall the works [3], [4] and [9]. Further, we study the behavior of quasi-Hadamard product on the class $T_0(p)$.

**Theorem 2.3.** Let the functions $f_i(z)$ belong to the classes $\beta - UST_0(p, q, \alpha_i), i = 1, 2, 3, \ldots, m$ and let the functions $g_j(z)$ belong to the classes $\beta - UCV_0(p, q, \gamma_j), j = 1, 2, 3, \ldots, d$. Then the quasi-Hadamard product $f_1 \ast f_2 \ast f_3 \ast \ldots \ast f_m \ast g_1 \ast g_2 \ast g_3 \ast \ldots \ast g_d(z)$ belongs to the class $\beta - T_0(m + 2d - 1, p, q, \rho)$, where

$$\rho = \max\{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_m, \gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_d\}.$$

**Proof.** For the sake of simplicity we will prove only the case when $m = d = 1$ and $\alpha_i = \gamma_j = \alpha$, the generalization being obtained easily. Hence we have to prove that if $f \in \beta - UST_0(p, q, \alpha), g \in \beta - UCV_0(p, q, \alpha)$ then $f \ast g \in \beta - T_0(2, p, q, \alpha)$. Since $f \in \beta - UST_0(p, q, \alpha)$ by the Theorem 2.1 we have:

$$\sum_{n=1}^{\infty} [(n + p - q - \alpha) + \beta(n + p - q - 1)] \cdot \delta(n + p, q) \cdot a_{n+p}$$

$$\leq [(p - q - \alpha) + \beta(p - q - 1)] \cdot \delta(p, q) \cdot a_p,$$

From this inequality we obtain further

$$a_{n+p} \leq \left[\frac{[(p - q - \alpha) + \beta(p - q - 1)] \cdot \delta(p, q)}{[(n + p - q - \alpha) + \beta(n + p - q - 1)] \cdot \delta(n + p, q)}\right] \cdot a_p. \quad (2.7)$$

We denote by $H(\alpha)$ the function

$$H(\alpha) = \frac{[(p - q - \alpha) + \beta(p - q - 1)]}{[(n + p - q - \alpha) + \beta(n + p - q - 1)]}.$$

Since $H(\alpha)$ is a decreasing function and $\frac{\delta(p, q)}{\delta(n + p, q)} \leq 1$, we obtain from (2.7), the following evaluation:

$$a_{n+p} \leq \left[\frac{[(p - q) + \beta(p - q - 1)]}{[(n + p - q) + \beta(n + p - q - 1)]}\right] \cdot a_p.$$

Now we denote by $G(\beta)$ the function
Corollary 2.2. Let the functions \( g_i \) belong to the classes \( \beta - UST_0(p, q, \alpha_i), i = 1, m \). Then the quasi Hadamard product \( f_1 \ast f_2 \ast \cdots \ast f_m \) belongs to the class \( \beta - T_0(m - 1, p, q, \rho) \), \( \rho = \max \{\alpha_1, \ldots, \alpha_m\} \).

Corollary 2.2. Let the functions \( g_j \) belong to the classes \( \beta - UCV_0(p, q, \gamma_j), j = 1, d \). Then the quasi Hadamard product \( g_1 \ast g_2 \ast \cdots \ast g_d \) belongs to the class \( \beta - T_0(2d - 1, p, q, \rho) \), \( \rho = \max \{\gamma_1, \ldots, \gamma_d\} \).

Remark 2.3. i) If in Theorem 2.3 and in Corollaries 2.1 and 2.2 we take \( \beta = 0 \) we obtain the same results with El-Ashwah et al., given in [3].

ii) If in Theorem 2.3 and in Corollaries 2.1 and 2.2 we take \( p = 1 \) and \( q = 0 \) we obtain the same results as the results given by Frasin in [4].
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