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Quasi-Hadamard product of some uniformly analytic and p-valent functions with negative coefficients

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ABSTRACT. In this paper we study the quasi-Hadamard product between some p-valent and uniformly analytic functions with negative coefficients defined in connection with uniformly starlikeness and uniformly convexity.

1. INTRODUCTION

Let $T_0(p)$ denote the class of functions of the form:

$$f(z) = a_p z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, (a_{p+n} \geq 0; p \in \mathbb{N} = \{1, 2, ...\}, a_p > 0)$$

which are analytic and p-valent in the open unit disc $U = \{z : z \in \mathbb{C} \; and \; |z| < 1\}$. By $T(p)$ we will denote the class $T_0(p)$ with $a_p = 1$.

Also let $S(p, q, \alpha)$ and $C(p, q, \alpha)$ denote two subclasses of $T(p)$ defined as:

$$S(p, q, \alpha) = \left\{ f(z) \in T(p) : \text{Re} \left\{ \frac{z f^{(1+q)}(z)}{f^{(q)}(z)} \right\} > \alpha \right. ,$$

$$\left. (z \in U; 0 \leq \alpha < p - q; p \in \mathbb{N}; p > q; q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}) \right\}$$

and

$$C(p, q, \alpha) = \left\{ f(z) \in T(p) : \text{Re} \left\{ 1 + \frac{z f^{(2+q)}(z)}{f^{(1+q)}(z)} \right\} > \alpha \right. ,$$

$$\left. (z \in U; 0 \leq \alpha < p - q; p \in \mathbb{N}; p > q; q \in \mathbb{N}_0) \right\}$$

These classes were investigated by Chen et al. in [2]. We can notice that:
(i) $S(p, 0, \alpha) = T^*(p, \alpha)$, is the class of p-valently starlike functions of order $\alpha, 0 \leq \alpha < p$;
(ii) C(p,0,α) $= C(p, \alpha)$, is the class of p-valently convex functions of order $\alpha, 0 \leq \alpha < p$.
These last particular classes, $T^*(p, \alpha)$ and $C(p, \alpha)$, were studied by Owa in [7], Sălăgean et al. in [8] and Sekine in [9].

For the classes $S(p, q, \alpha)$ and $C(p, q, \alpha)$, Chen et al. obtained the following results:

Lemma 1.1. [2] *A function $f(z) \in T(p)$ is in the class $S(p, q, \alpha)$ if and only if*

$$\sum_{n=1}^{\infty} (n + p - q - \alpha)\delta(n + p, q) a_{p+n} \leq (p - q - \alpha)\delta(p, q)$$

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$$(0 \le \alpha < p-q; p \in \mathbb{N}; p > q; q \in \mathbb{N}_0),$$

where

$$\delta(p,q) = \frac{p!}{(p-q)!} = \begin{cases} p(p-1)...(p-q+1) & (q \neq 0) \\ 1 & (q = 0). \end{cases}$$

Lemma 1.2. [2] *A function $f(z) \in T(p)$ is in the class $C(p,q,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty} \binom{n+p-q}{p-q} (n+p-q-\alpha)\delta(n+p,q)a_{p+n} \le (p-q-\alpha)\delta(p,q).$$

$$(0 \le \alpha < p-q; p \in \mathbb{N}; p > q; q \in \mathbb{N}_0),$$

where

$$\delta(p,q) = \frac{p!}{(p-q)!} = \begin{cases} p(p-1)...(p-q+1) & (q \neq 0) \\ 1 & (q = 0). \end{cases}$$

In order to present the main results of this paper, we introduce two new classes of p-valent functions with negative coefficients defined by some conditions related to uniformly starlikeness and uniformly convexity, in the following manner:

$$\beta - UST_0(p,q,\alpha) = \left\{ f \in T_0(p) : \mathrm{Re}\left\{ \frac{zf^{(1+q)}(z)}{f^{(q)}(z)} \right\} \ge \beta \left| \frac{zf^{(1+q)}(z)}{f^{(q)}(z)} - 1 \right| + \alpha \right\},$$

$$\left(z \in U; 0 \le \alpha < p-q; p \in \mathbb{N}; p > q; q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \beta \ge 0\right)$$

and

$$\beta - UCV_0(p,q,\alpha) = \left\{ f \in T_0(p) : \mathrm{Re}\left\{ 1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} \right\} \ge \beta \left| \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} \right| + \alpha \right\},$$

$$\left(z \in U; 0 \le \alpha < p-q; p \in \mathbb{N}; p > q; q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \beta \ge 0\right).$$

Remark 1.1. i) For $\beta = 0$ we have $\beta - UST_0(p,q,\alpha) = S_0(p,q,\alpha)$ and $\beta - UCV_0(p,q,\alpha) = C_0(p,q,\alpha)$, analogous to $S(p,q,\alpha)$ and $C(p,q,\alpha)$, but defined for functions from $T_0(p)$, the classes considered and investigated by El Ashwah et al. in [3] and for $\beta = 0$ and $a_p = 1$ we get the classes $S(p,q,\alpha)$ and $C(p,q,\alpha)$, studied by Chen et al. in [2].

ii) For $p = 1,\ q = 0$ the classes $\beta - UST_0(\alpha)$ and $\beta - UCV_0(\alpha)$ are the classes of β-uniformly starlike functions of order α and β-uniformly convex functions of order α introduced by Bharati et al. for $a_p = 1$ in [1] and by Frasin for $a_p \neq 1$ in [4].

iii) For $p = 1,\ q = 0,\ \alpha = 0$ we get the classes of $\beta - UCV$ and $\beta - UST$ of β uniformly convex and starlike functions introduced by Kanas and Wisniowska in the papers [5] and [6].

iv) For $p = 1,\ q = 0,\ \beta = 0$ we get the well known classes of convex respectively starlike functions of order α.

v) The new introduced classes are nontrivial generalizations of the above mentioned classes $\left(\text{consider for example, the function } f(z) = z^2 - \frac{1}{6}z^3\right)$.

2. Main results

First, we obtain coefficients estimates for the classes $\beta{-}UST_0(p,q,\alpha)$ and $\beta{-}UCV_0(p,q,\alpha)$.

Theorem 2.1. *A function $f \in \beta - UST_0(p,q,\alpha)$ satisfies the inequality*

(2.1)
$$\sum_{n=1}^{\infty}[(n+p-q-\alpha)+\beta(n+p-q-1)]\cdot\delta(n+p,q)\cdot a_{n+p} \le [(p-q-\alpha)+\beta(p-q-1)]\cdot\delta(p,q)\cdot a_p,$$

where

$$\delta(p,q)=\frac{p!}{(p-q)!}=\begin{cases} p(p-1)...(p-q+1) & (q\neq 0)\\ 1 & (q=0)\,.\end{cases}$$

Proof. For the begining we prove (2.1) for $a_p=1$. Since $f\in\beta - UST_0(p,q,\alpha)$ we get

$$\text{Re}\left\{\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\right\} \ge \beta\left|\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}-1\right|+\alpha \tag{2.2}$$

$$\left(z\in U; 0\le\alpha<p-q; p\in\mathbb{N}; p>q; q\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\ \beta\ge 0\right).$$

Since $|w|\ge -\text{Re}w$ from (2.2) we obtain

$$\text{Re}\left\{\frac{zf^{(1+q)}(z)}{f^{(q)}(z)}\right\}\ge\frac{\alpha+\beta}{1+\beta}$$

We can notice that $0\le\gamma=\dfrac{\alpha+\beta}{1+\beta}<p-q$, hence we can say that $f\in S(p,q,\gamma)$.

From Lemma 1.1 it follows that

$$\sum_{n=1}^{\infty}(n+p-q-\gamma)\delta(n+p,q)a_{p+n}\le(p-q-\gamma)\delta(p,q) \tag{2.3}$$

$$(0\le\gamma<p-q; p\in\mathbb{N}; p>q; q\in\mathbb{N}_0)\,.$$

Now if we put $\gamma=\dfrac{\alpha+\beta}{1+\beta}$ the inequality (2.3) is equivalent with (2.1) (for $a_p=1$) so the proof is done for $a_p=1$.

When $a_p\neq 1$ we have $f(z)=a_p\cdot g(z)$ with $g(z)$ of the form

$$g(z)=z^p-\sum_{n=1}^{\infty}\frac{a_{p+n}}{a_p}z^{p+n}$$

satisfying the requested inequality hence f satisfies (2.1) and the proof is complete. □

Theorem 2.2. *A function $f\in\beta-UCV_0(p,q,\alpha)$ satisfies the inequality*

$$\sum_{n=1}^{\infty}\binom{n+p-q}{p-q}[(n+p-q-\alpha)+\beta(n+p-q-1)]\cdot\delta(n+p,q)\cdot a_{n+p}$$

$$\le[(p-q-\alpha)+\beta(p-q-1)]\cdot\delta(p,q)\cdot a_p, \tag{2.4}$$

where

$$\delta(p,q)=\frac{p!}{(p-q)!}=\begin{cases} p(p-1)...(p-q+1) & (q\neq 0)\\ 1 & (q=0)\,.\end{cases}$$

Proof. From the same reasons as in Theorem 2.1 it is sufficient to prove the inequality (2.4) for $a_p = 1$. Let $f \in \beta - UCV_0(p, q, \alpha)$, with $a_p = 1$. Then we have

$$\operatorname{Re}\left\{1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)}\right\} \geq \beta \left|\frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)}\right| + \alpha.$$

Further we derive

$$\operatorname{Re}\left\{\frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)}\right\} \geq \frac{\alpha - 1}{1+\beta}.$$

and if we denote $\gamma - 1 = \dfrac{\alpha - 1}{1+\beta}$, $0 \leq \gamma < p - q$, we have

$$\operatorname{Re}\left\{1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)}\right\} \geq \gamma.$$

Here we can recognize that $f \in C(p, q, \gamma)$. Then from Lemma 1.2 it follows the inequality

$$\sum_{n=1}^{\infty} \left(\frac{n+p-q}{p-q}\right) (n+p-q-\gamma)\delta(n+p, q)a_{p+n} \leq (p-q-\gamma)\delta(p, q)\,. \tag{2.5}$$

If we take in (2.5), $\gamma = \dfrac{\beta + \alpha}{\beta + 1}$, then we obtain the inequality (2.4), for $a_p = 1$ and the proof is complete. □

In what follows we introduce a subclass of the class $T_0(p)$. We say that a function $f(z)$ from $T_0(p)$ belongs to the class $\beta - T_0(k, p, q, \alpha)$ if and only if

$$\sum_{n=1}^{\infty} \left(\frac{n+p-q}{p-q}\right)^k \left[(n+p-q-\alpha) + \beta(n+p-q-1)\right] \delta(n+p, q)a_{p+n}$$

$$\leq \left[(p-q-\alpha) + \beta(p-q-1)\right] \delta(p, q)a_p, \tag{2.6}$$

$0 \leq \alpha < p - q,\ p \in \mathbb{N}, q \in \mathbb{N} \cup \{0\}, \beta \geq 0$, where k is any fixed non-negative real number.

Remark 2.2. i) We can notice that the class $\beta - T_0(k, p, q, \alpha)$ is nonempty since the function of the form

$$f(z) = a_p z^p - \sum_{n=1}^{\infty} \frac{\left[(p-q-\alpha) + \beta(p-q-1)\right] \delta(p, q)a_p}{\left(\frac{n+p-q}{p-q}\right)^k \left[(n+p-q-\alpha) + \beta(n+p-q-1)\right] \delta(n+p, q)} \cdot \lambda_{p+n} z^{p+n},$$

where $a_p > 0$, $\lambda_{p+n} > 0$ and $\sum\limits_{n=1}^{\infty} \lambda_{p+n} \leq 1$, satisfies the inequality (2.6).

ii) For $k > c \geq 0$, we have that

$$\beta - T_0(k, p, q, \alpha) \subset \beta - T_0(c, p, q, \alpha)\,.$$

iii) Also we have the inclusion relations

$$\beta - UST_0(p, q, \alpha) \subset \beta - T_0(0, p, q, \alpha)$$

$$\beta - UCV_0(p, q, \alpha) \subset \beta - T_0(1, p, q, \alpha).$$

Now, in order to study the quasi-Hadamard product, let the functions f_i and g_j from the class $T_0(p)$ be of the forms:

$$f_i(z) = a_{p,i}z^p - \sum_{n=1}^{\infty} a_{p+n,i}z^{p+n} \qquad (a_{p,i} > 0; a_{p+n,i} \geq 0)$$

$$g_j(z) = b_{p,j}z^p - \sum_{n=1}^{\infty} b_{p+n,j}z^{p+n} \qquad (b_{p,j} > 0; b_{p+n,j} \geq 0)\,,$$

The quasi-Hadamard product $f_i * g_j(z)$ of the functions $f_i(z)$ and $g_j(z)$ is defined by

$$f_i * g_j(z) = a_{p,i}b_{p,j}z^p - \sum_{n=1}^{\infty} a_{p+n,i}b_{p+n,j}z^{p+n} \qquad (i, j = 1, 2, 3, ...)\,.$$

The quasi-Hadamard product (almost a Hadamard product excepting that the minus sign is kept in front of the sum) has been studied before in various papers, among we recall the works [3], [4] and [9]. Further, we study the behavior of quasi-Hadamard product on the class $T_0(p)$.

Theorem 2.3. *Let the functions $f_i(z)$ belong to the classes $\beta - UST_0(p, q, \alpha_i), i = 1, 2, 3, ..., m$ and let the functions $g_j(z)$ belong to the classes $\beta - UCV_0(p, q, \gamma_j), j = 1, 2, 3, ..., d$. Then the quasi-Hadamard product $f_1 * f_2 * f_3 * ... * f_m * g_1 * g_2 * g_3 * ... * g_d(z)$ belongs to the class $\beta - T_0(m + 2d - 1, p, q, \rho)$, where*

$$\rho = \max\{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_m, \gamma_1, \gamma_2, \gamma_3, ..., \gamma_d\}\,.$$

Proof. For the sake of simplicity we will prove only the case when $m = d = 1$ and $\alpha_i = \gamma_j = \alpha$, the generalization being obtained easily. Hence we have to prove that if $f \in \beta - UST_0(p, q, \alpha)$, $g \in \beta - UCV_0(p, q, \alpha)$ then $f * g \in \beta - T_0(2, p, q, \alpha)$. Since $f \in \beta - UST_0(p, q, \alpha)$ by the Theorem 2.1 we have:

$$\sum_{n=1}^{\infty} [(n + p - q - \alpha) + \beta(n + p - q - 1)] \cdot \delta(n + p, q) \cdot a_{n+p}$$
$$\leq [(p - q - \alpha) + \beta(p - q - 1)] \cdot \delta(p, q) \cdot a_p,$$

From this inequality we obtain further

$$a_{n+p} \leq \left[\frac{[(p - q - \alpha) + \beta(p - q - 1)] \cdot \delta(p, q)}{[(n + p - q - \alpha) + \beta(n + p - q - 1)] \cdot \delta(n + p, q)}\right] \cdot a_p. \tag{2.7}$$

We denote by $H(\alpha)$ the function

$$H(\alpha) = \frac{[(p - q - \alpha) + \beta(p - q - 1)]}{[(n + p - q - \alpha) + \beta(n + p - q - 1)]}.$$

Since $H(\alpha)$ is a decreasing function and $\dfrac{\delta(p, q)}{\delta(n + p, q)} \leq 1$, we obtain from (2.7), the following evaluation:

$$a_{n+p} \leq \left[\frac{[(p - q) + \beta(p - q - 1)]}{[(n + p - q) + \beta(n + p - q - 1)]}\right] \cdot a_p.$$

Now we denote by $G(\beta)$ the function

$$G(\beta)=\frac{[(p-q)+\beta(p-q-1)]}{[(n+p-q)+\beta(n+p-q-1)]}.$$

which is also a decreasing function, hence

$$a_{n+p}\leq\frac{p-q}{n+p-q}\cdot a_p. \tag{2.8}$$

On the other side, from the definiton of the class $\beta-T_0(2,p,q,\alpha)$ it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{n+p-q}{p-q}\right)^2[(n+p-q-\alpha)+\beta(n+p-q-1)]\cdot$$
$$\cdot\delta(n+p,q)a_{p+n}b_{p+n}$$
$$\leq[(p-q-\alpha)+\beta(p-q-1)]\,\delta(p,q)a_pb_p. \tag{2.9}$$

We evaluate the left term of the inequality (2.9) by using (2.8) and we get

$$\sum_{n=1}^{\infty}\left(\frac{n+p-q}{p-q}\right)^2[(n+p-q-\alpha)+\beta(n+p-q-1)]\cdot$$
$$\cdot\delta(n+p,q)a_{p+n}b_{p+n}\leq$$
$$\sum_{n=1}^{\infty}\left(\frac{n+p-q}{p-q}\right)^2[(n+p-q-\alpha)+\beta(n+p-q-1)]\cdot$$
$$\cdot\delta(n+p,q)\left[\frac{p-q}{n+p-q}\cdot a_p\right]b_{p+n}. \tag{2.10}$$

Further in the last inequality we apply the fact that $g\in\beta-UCV_0(p,q,\alpha)$, namely

$$\sum_{n=1}^{\infty}\left(\frac{n+p-q}{p-q}\right)[(n+p-q-\alpha)+\beta(n+p-q-1)]\,\delta(n+p,q)b_{p+n}$$
$$\leq[(p-q-\alpha)+\beta(p-q-1)]\,\delta(p,q)b_p. \tag{2.11}$$

If we put (2.11) in (2.10) the proof is complete. □

If in Theorem 2.3 we consider the functions only from $\beta-UST_0(p,q,\alpha)$ or only from $\beta-UCV_0(p,q,\alpha)$ we obtain next corollaries:

Corollary 2.1. *Let the functions f_i belong to the classes $\beta-UST_0(p,q,\alpha_i)$, $i=\overline{1,m}$. Then the quasi Hadamard product $f_1*f_2*...*f_m$ belongs to the class $\beta-T_0(m-1,p,q,\rho)$, $\rho=\max\{\alpha_1,...,\alpha_m\}$.*

Corollary 2.2. *Let the functions g_j belong to the classes $\beta-UCV_0(p,q,\gamma_j)$, $j=\overline{1,d}$. Then the quasi Hadamard product $g_1*g_2*...*g_d$ belongs to the class $\beta-T_0(2d-1,p,q,\rho)$, $\rho=\max\{\gamma_1,...,\gamma_d\}$.*

Remark 2.3. i) If in Theorem 2.3 and in Corollaries 2.1 and 2.2 we take $\beta=0$ we obtain the same results with El-Ashwah et al., given in [3].

ii) If in Theorem 2.3 and in Corollaries 2.1 and 2.2 we take $p=1$ and $q=0$ we obtain the same results as the results given by Frasin in [4].

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