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A CERTAIN CONVOLUTION APPROACH FOR SUBCLASSES OF UNIVALENT HARMONIC FUNCTIONS

R. M. EL-ASHWAH* AND M. K. AOUF

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Abstract. In the present paper we study convolution properties for subclasses of univalent harmonic functions in the open unit disc and obtain some basic properties such as coefficient characterization and extreme points.
Keywords: Analytic, harmonic, convolution.
MSC(2010): Primary: 30C45.

1. Introduction

A continuous function $f = u+iv$ is a complex valued harmonic function in a simply connected complex domain $D \subset C$ if both u and v are real harmonic in D. It was shown by Clunie and Sheil-Small [5] that such harmonic function can be represented by $f = h+\overline{g}$, where h and g are analytic in D. Also, a necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $\left|h^{'}(z)\right| > \left|g^{'}(z)\right|$, (see also, [2, 9, 10] and [20]).

Denote by H the family of functions $f = h + \overline{g}$, which are harmonic univalent and sense-preserving in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ with normalization $f(0) = h(0) = f_z'(0) - 1 = 0$ and h, g are of the form:

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ g(z) = \sum_{n=1}^{\infty} b_n z^n \ \ (0 \le |b_1| < 1; z \in U). \tag{1.1}$$

Also denote by TH the subfamily of H consisting of harmonic functions $f = h + g$ of the form:

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n, \ g(z) = \sum_{n=1}^{\infty} b_n z^n \ \ (a_n, b_n \ge 0; 0 \le |b_1| < 1; z \in U). \tag{1.2}$$

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Let

$$\phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n \quad (\phi_n \geq 0; n \in \mathbb{N} = \{1, 2, ...\}), \tag{1.3}$$

be a given analytic function in U, then

$$\left(\phi(z) + \overline{\phi(z)}\right) * \left(h(z) + \overline{g(z)}\right) = z + \sum_{n=2}^{\infty} \phi_n a_n z^n + \sum_{n=1}^{\infty} \phi_n b_n \overline{z}^n$$
$$(\phi_1 = 1; 0 \leq |b_1| < 1; z \in U), \tag{1.4}$$

where $*$ is the convolution operation.

Later on, variouse subclasses of H have been introduced and studied by several authors (see e. g. [2, 11–14]). We shall show in this note that these subclasses are special cases of the general class $H^0(\phi, \lambda, \sigma, \alpha)$ giving in the following definition.

Definition 1.1. *Let $\phi(z)$ be given by (1.3). A harmonic function $f = h + \overline{g} \in H$ where h and g are given by (1.1) belongs to the class $H^0(\phi, \lambda, \sigma, \alpha)$ if*

$$\operatorname{Re}\left\{\frac{\left(z(h*\phi)'(z) - \sigma\overline{z(g*\phi)'(z)}\right)}{(1-\lambda)\left((h*\phi)(z) + \sigma\overline{(g*\phi)(z)}\right) + \lambda\left(z(h*\phi)'(z) - \sigma\overline{z(g*\phi)'(z)}\right)}\right\} > \alpha$$

$$(0 \leq \alpha < 1; 0 \leq \lambda \leq 1; |\sigma| \leq 1; z \in U), \tag{1.5}$$

We note that

(i) $H^0\left(H_q^s[\alpha_1], \lambda, 1, \alpha\right) = R_H^{s,q}([\alpha_1], \lambda, \alpha)$ (where $H_q^s[\alpha_1]$ is the Dziok-Srivastava operator [6]) (see Murugusundaramoorthy et al. [12]);

(ii) $H^0\left(H_q^s[\alpha_1], 0, 1, \alpha\right) = R_H^{s,q}([\alpha_1], \alpha)$ (where $H_q^s[\alpha_1]$ is the Dziok-Srivastava operator [6]) (see Al-kharsani and Al-Khal [2]);

(iii) $H^0\left(\frac{z}{(1-z)}, \lambda, 1, \alpha\right) = S_H^*(\lambda, \alpha)$ (see Ozturk et al. [14]);

(iv) $H^0\left(\frac{z}{(1-z)}, 0, 1, \alpha\right) = S_H^*(\alpha)$ and $H^0\left(\frac{z}{(1-z)^2}, 0, -1, \alpha\right) = K_H(\alpha)$ (see Jahangiri [10]);

(v) $H^0\left(\frac{z}{(1-z)}, 0, 1, 0\right) = S_H^*$ and $H^0\left(\frac{z}{(1-z)^2}, 0, -1, 0\right) = K_H$ (see Silverman [19]).

Now we denote by $HP^0(\phi, \lambda, \sigma, k, \alpha)$ a general class of various subclasses of harmonic parabolic starlike functions introduced and studied by several authors (see e. g. [8, 16, 17] and [21]).

Definition 1.2. *Let $\phi(z)$ be given by (1.3). A harmonic function $f = h + \overline{g} \in H$ belongs to the class $HP^0(\phi, \lambda, \sigma, k, \alpha)$ if*

$$\operatorname{Re}\left\{\frac{\left(1+ke^{i\gamma}\right)\left(z(h*\phi)'(z) - \sigma\overline{z(g*\phi)'(z)}\right)}{(1-\lambda)\left((h*\phi)(z) + \sigma\overline{(g*\phi)(z)}\right) + \lambda\left(z(h*\phi)'(z) - \sigma\overline{z(g*\phi)'(z)}\right)} - ke^{i\gamma}\right\} > \alpha$$

(1.6) $$\left(0 \le \alpha < 1; 0 \le \lambda \le 1; |\sigma| \le 1; k \ge 0; \gamma \text{ real}; z \in U\right).$$

We note that

(i) $HP^0\left(z+\sum\limits_{n=2}^{\infty} n^m z^n, \lambda, (-1)^m, 1, \alpha\right) = S_H^*\ (m+1, m, \lambda, \alpha)$ (see Sudharasan et al. [21]);

(ii) $HP^0\left(\frac{z}{1-z}, \lambda, 1, 0, \alpha\right) = S_H^*, (\lambda, \alpha)$ (see Ozturk et al. [14]));

(iii) $HP^0\left(\frac{z}{1-z}, 0, 1, 1, \alpha\right) = G_H(\alpha)$ (see Rosy et al. [16]).

In this paper, we obtain necessary and sufficient convolution conditions for the classes $H^0(\phi, \lambda, \sigma, \alpha)$ and $HP^0(\phi, \lambda, \sigma, k, \alpha)$, sufficient coefficient bounds for functions in these two classes, these sufficient coefficient conditions are shown to be also necessary for functions with negative coefficients. Finally we determined growth estimates and extreme points for the class $TH^0(\phi, \lambda, \sigma, \alpha) = H^0(\phi, \lambda, \sigma, \alpha) \cap TH$. The class $TH^0(\phi, \lambda, \sigma, \alpha)$ includes as special cases several subclasses studied in [9] and [19].

2. Main results

We now derive the convolution characterization for functions in the class $H^0(\phi, \lambda, \sigma, \alpha)$.

Theorem 2.1. *Let $f = h + \bar{g} \in H$ and $\phi(z)$ be given by (1.3), then $f \in H^0(\phi, \lambda, \sigma, \alpha)$ if and only if*

$$(h*\phi)*\left[\frac{z+\frac{(1-\lambda)(x+2\alpha-1)}{2(1-\alpha)}z^2}{(1-z)^2}\right] - \sigma\overline{(g*\phi)}*\left[\frac{\frac{x+\alpha-\lambda(x+2\alpha-1)}{1-\alpha}\overline{z} - \frac{(1-\lambda)(x+2\alpha-1)}{2(1-\alpha)}\bar{z}^2}{(1-\bar{z})^2}\right] \ne 0$$

(2.1) $$(|x| = 1; |z| \ne 0).$$

Proof. A necessary and sufficient condition for $f = h + \bar{g}$ to be in the class $H^0(\phi, \lambda, \sigma, \alpha)$, with h, g of the form (1.1), is given by (1.5). Since

$$\frac{\left(z(h*\phi)'(z) - \sigma\overline{z(g*\phi)'(z)}\right)}{(1-\lambda)\left((h*\phi)(z) + \sigma\overline{(g*\phi)(z)}\right) + \lambda\left(z(h*\phi)'(z) - \sigma\overline{z(g*\phi)'(z)}\right)} = 1$$

at $z = 0$, the condition (1.5) is equivalent to

$$\frac{1}{1-\alpha}\left\{\frac{\left(z(h*\phi)'(z) - \sigma\overline{z(g*\phi)'(z)}\right)}{(1-\lambda)\left((h*\phi)(z) + \sigma\overline{(g*\phi)(z)}\right) + \lambda\left(z(h*\phi)'(z) - \sigma\overline{z(g*\phi)'(z)}\right)} - \alpha\right\} \ne \frac{x-1}{x+1}$$

(2.2) $$(|x| = 1; x \ne -1; 0 < |z| < 1).$$

By a simple algebraic manipulation, (2.2) yields

$$\begin{aligned}
0 \neq\ & (x+1)\left[z\left(h*\phi\right)'(z)-\sigma z\overline{(g*\phi)'}(z)\right] \\
& -\alpha(x+1)\left[(1-\lambda)\left(\left(h*\phi\right)(z)+\sigma\overline{\left(g*\phi\right)(z)}\right)\right. \\
& \left.+\lambda\left(z\left(h*\phi\right)'(z)-\sigma\overline{z\left(g*\phi\right)'(z)}\right)\right] \\
& -(x-1)(1-\alpha)\left[(1-\lambda)\left(\left(h*\phi\right)(z)+\sigma\overline{\left(g*\phi\right)(z)}\right)\right. \\
& \left.+\lambda\left(z\left(h*\phi\right)'(z)-\sigma\overline{z\left(g*\phi\right)'(z)}\right)\right] \\
=\ & (h*\phi)*\left[\frac{2(1-\alpha)z+(1-\lambda)\left(x+2\alpha-1\right)z^2}{(1-z)^2}\right] \\
& -\sigma\overline{(g*\phi)}*\left[\frac{2[x+\alpha-\lambda(x+2\alpha-1)]\overline{z}-(1-\lambda)\left(x+2\alpha-1\right)\overline{z}^2}{(1-\overline{z})^2}\right].
\end{aligned}$$

The latter condition together with (1.5) establishes the result (2.1) for all $|x|=1$. □

Remark 2.2. *Taking $\lambda=0$ and $b_1=0$ in Theorem 2.1 we obtain the result obtained by Ali et al. [3, Theorem 2];*

(ii) Taking $\lambda=0,\sigma=1,\alpha=0$ and $\phi=\dfrac{z}{1-z}$, in Theorem 2.1 we obtain the result obtained by Ahuja et al. [1, Theorem 2.6];

(ii) Taking $\lambda=0,\sigma=1,\alpha=0$ and $\phi=\dfrac{z}{(1-z)^2}$, in Theorem 2.1 we obtain the result obtained by Ahuja et al. [1, Theorem 2.8].

Necessary coefficient conditions for harmonic starlike functions were obtained in [5] (see also [18]).

Using the convolution characterization, we now derive a sufficient condition for harmonic functions to belong to the class $H^0(\phi,\lambda,\sigma,\alpha)$.

Theorem 2.3. *Let $f=h+\bar{g}\in H$ and $\phi(z)$ be given by (1.3), then $f\in H^0(\phi,\lambda,\sigma,\alpha)$ if*

$$\sum_{n=2}^{\infty}\frac{n(1-\alpha\lambda)-\alpha(1-\lambda)}{(1-\alpha)}\left|a_n\right|\left|\phi_n\right|+|\sigma|\sum_{n=1}^{\infty}\frac{n(1-\alpha\lambda)+\alpha(1-\lambda)}{(1-\alpha)}\left|b_n\right|\left|\phi_n\right|\leq 1.$$

Proof. For h and g given by (1.1), (2.1) gives

$$\left|(h*\phi)*\left[\frac{z+\frac{(1-\lambda)(x+2\alpha-1)}{2(1-\alpha)}z^2}{(1-z)^2}\right]-\sigma\overline{(g*\phi)}*\left[\frac{\frac{x+\alpha-\lambda(x+2\alpha-1)}{1-\alpha}\bar{z}-\frac{(1-\lambda)(x+2\alpha-1)}{2(1-\alpha)}\bar{z}^2}{(1-\bar{z})^2}\right]\right|$$
$$=\left|z+\sum_{n=2}^{\infty}\left\{n+(n-1)\frac{(1-\lambda)(x+2\alpha-1)}{2(1-\alpha)}\right\}a_n\phi_n z^n\right.$$
$$\left.-\sigma\sum_{n=1}^{\infty}\left\{n\frac{x+\alpha-\lambda(x+2\alpha-1)}{1-\alpha}-(n-1)\frac{(1-\lambda)(x+2\alpha-1)}{2(1-\alpha)}\right\}b_n\phi_n z^n\right|$$
$$\geq |z|\left[1-\sum_{n=2}^{\infty}\frac{n(1-\alpha\lambda)-\alpha(1-\lambda)}{(1-\alpha)}|a_n||\phi_n|-|\sigma|\sum_{n=1}^{\infty}\frac{n(1-\alpha\lambda)+\alpha(1-\lambda)}{(1-\alpha)}|b_n||\phi_n|\right].$$

The last expression is non-negative by hypothesis, and hence by Theorem 2.1, it follows that $f\in H_p^0(\phi,\lambda,\sigma,\alpha)$. □

Remark 2.4. *i) Taking $\lambda=0$ and $b_1=0$ in Theorem 2.3 we obtain the result obtained by Ali et al. [3, Theorem 3];*

(ii) Taking $\lambda=0,\sigma=1,\alpha=0$ and $\phi=\frac{z}{1-z}$, in Theorem 2.3 we obtain the result obtained by Ahuja et al. [1, Corollary 2.7];

(iii) Taking $\lambda=0,,\sigma=1,\alpha=0$ and $\phi=\frac{z}{(1-z)^2}$, in Theorem 2.3 we obtain the result obtained by Ahuja et al. [1, Corollary 2.9].

Theorem 2.5. *Let $f=h+\bar{g}\in H$ and $\phi(z)$ be given by (1.3), then $f\in HP^0(\phi,\lambda,\sigma,k,\alpha)$ if and only if*

$$(h*\phi)*\left[\frac{z+\frac{(1-\lambda)\left[(x+1)ke^{i\gamma}+x+2\alpha-1\right]}{2(1-\alpha)}z^2}{(1-z)^2}\right]$$
$$-\sigma\overline{(g*\phi)}*\left[\frac{\frac{(x+1)ke^{i\gamma}+x+\alpha-\lambda((x+1)ke^{i\gamma}+x+2\alpha-1)}{1-\alpha}\bar{z}-\frac{(1-\lambda)\left[(x+1)ke^{i\gamma}+x+2\alpha-1\right]}{2(1-\alpha)}\bar{z}^2}{(1-\bar{z})^2}\right]$$
$$\neq 0\quad(|x|=1;|z|\neq 0).$$

Proof. A necessary and sufficient condition for $f=h+\bar{g}$ to be in the class $HP^0(\phi,\lambda,\sigma,k,\alpha)$, with h,g of the form (1.1), is given by (1.6). Since

$$\frac{\left(1+ke^{i\gamma}\right)\left(z(h*\phi)'(z)-\sigma\overline{z(g*\phi)'(z)}\right)}{(1-\lambda)\left((h*\phi)(z)+\sigma\overline{(g*\phi)(z)}\right)+\lambda\left(z(h*\phi)'(z)-\sigma\overline{z(g*\phi)'(z)}\right)}-ke^{i\gamma}=1$$

at $z=0$, the condition (1.6) is equivalent to

$$\frac{1}{1-\alpha}\left\{\frac{\left(1+ke^{i\gamma}\right)\left(z(h*\phi)'(z)-\sigma\overline{z(g*\phi)'(z)}\right)}{(1-\lambda)\left((h*\phi)(z)+\sigma\overline{(g*\phi)(z)}\right)+\lambda\left(z(h*\phi)'(z)-\sigma\overline{z(g*\phi)'(z)}\right)}-ke^{i\gamma}-\alpha\right\}\neq\frac{x-1}{x+1}$$

$$(|x| = 1; x \neq -1; z \neq 0). \tag{2.3}$$

By a simple algebraic manipulation similar to that used in Theorem 2.1, we obtain the desired result. □

Proceeding similarly as in Theorem 2.3, the following sufficient condition for the class $HP^0(\phi, \lambda, \sigma, k, \alpha)$ is easily derived.

Theorem 2.6. *Let $f = h + \overline{g} \in H$ and $\phi(z)$ be given by (1.3), then $f \in HP^0(\phi, \lambda, \sigma, k, \alpha)$ if*

$$\sum_{n=2}^{\infty} \frac{n[(k+1) - \lambda(k+\alpha)] - (1-\lambda)(k+\alpha)}{(1-\alpha)} |a_n| |\phi_n| +$$

$$|\sigma| \sum_{n=1}^{\infty} \frac{n[(k+1) - \lambda(k+\alpha)] + (1-\lambda)(k+\alpha)}{(1-\alpha)} |b_n| |\phi_n| \leq 1.$$

Theorem 2.7. *Let $\phi(z)$ be given by (1.3) and $f = h + \overline{g}$ where h, g are given by (1.2). Then $f \in TH^0(\phi, \lambda, \sigma, \alpha)$ if and only if*

$$\sum_{n=2}^{\infty} \frac{n(1-\alpha\lambda) - \alpha(1-\lambda)}{(1-\alpha)} |a_n| |\phi_n| + |\sigma| \sum_{n=1}^{\infty} \frac{n(1-\alpha\lambda) + \alpha(1-\lambda)}{(1-\alpha)} |b_n| |\phi_n| \leq 1. \tag{2.4}$$

Proof. If f belongs to the class $TH^0(\phi, \lambda, \sigma, \alpha)$, then (1.5) is equivalent to

$$\operatorname{Re}\left\{ \frac{(1-\alpha)z - \sum_{n=2}^{\infty}[n(1-\alpha\lambda) - \alpha(1-\lambda)]a_n\phi_n z^n - \sigma \sum_{n=1}^{\infty}[n(1-\alpha\lambda) + \alpha(1-\lambda)]b_n\phi_n \overline{z}^n}{z - \sum_{n=2}^{\infty}[(1-\lambda) + n\lambda]a_n\phi_n z^n + \sigma \sum_{n=1}^{\infty}[(1-\lambda) - n\lambda]b_n\phi_n \overline{z}^n} \right\} > 0,$$

for $z \in U$. Letting $z \to 1^-$ through real values yields condition (2.4). □

Conversely, for h, g given by (1.2), we have

$$(h*\phi) * \left[\frac{z + \frac{(1-\lambda)(x+2\alpha-1)}{2(1-\alpha)} z^2}{(1-z)^2} \right] - \sigma \overline{(g*\phi)} * \left[\frac{\frac{x+\alpha-\lambda(x+2\alpha-1)}{1-\alpha}\overline{z} - \frac{(1-\lambda)(x+2\alpha-1)}{2(1-\alpha)}\overline{z}^2}{(1-\bar{z})^2} \right]$$

$$\geq |z| \left[1 - \sum_{n=2}^{\infty} \frac{n(1-\alpha\lambda) - \alpha(1-\lambda)}{(1-\alpha)} |a_n| |\phi_n| - |\sigma| \sum_{n=1}^{\infty} \frac{n(1-\alpha\lambda) + \alpha(1-\lambda)}{(1-\alpha)} |b_n| |\phi_n| \right]$$

which is non-negative by hypothesis, this completes the proof of Theorem 2.7.

Remark 2.8. *(i) Taking $\sigma = 1$ and $\phi(z) = z + \sum_{n=2}^{\infty} \Gamma_{n-1}(\alpha_1) z^n$, where*

$$\Gamma_{n-1}(\alpha_1) = \frac{(\alpha_1)_{n-1} \ldots (\alpha_q)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_s)_{n-1} (1)_{n-1}}$$

$$(q \leq s+1, \alpha_i \in \mathbb{C} (i = 1, 2, ..., q) \text{ and } \beta_j \in \mathbb{C} \backslash \mathbb{Z}_0^- (j = 1, 2, ...), \tag{2.5}$$

in Theorem 2.7 we obtain the result obtained by Murugusundaramoorthy et al. [13].

Corollary 2.9. *Let* $\phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n$ *with* $\phi_n \geq \phi_2 (n \geq 2)$, *and* $|\sigma| \geq \frac{2-\alpha(1+\lambda)}{2+\alpha(1-3\lambda)}$. *If* $f \in TH^0(\phi, \lambda, \sigma, \alpha)$ *satisfies* $\frac{2-\alpha(1+\lambda)}{2+\alpha(1-3\lambda)} \left(\frac{1+\alpha(1-\lambda)}{1-\alpha}\right) |b_1| < 1$, *then for* $|z| = r < 1$,

$$|f(z)| \leq (1+|b_1|)\, r + \frac{1-\alpha}{(2-\alpha(1+\lambda))\phi_2} \left(1 - \frac{1+\alpha(1-\lambda)}{1-\alpha} . \frac{2-\alpha(1+\lambda)}{2+\alpha(1-3\lambda)} |b_1| \right) r^2$$

and

$$|f(z)| \geq (1-|b_1|)\, r - \frac{1-\alpha}{(2-\alpha(1+\lambda))\phi_2} \left(1 - \frac{1+\alpha(1-\lambda)}{1-\alpha} . \frac{2-\alpha(1+\lambda)}{2+\alpha(1-3\lambda)} |b_1| \right) r^2.$$

The function

$$f(z) = z + b_1 \overline{z} + \frac{1-\alpha}{(2-\alpha(1+\lambda))\phi_2} \left(1 - \frac{1+\alpha(1-\lambda)}{1-\alpha} . \frac{2-\alpha(1+\lambda)}{2+\alpha(1-3\lambda)} b_1 \right) \overline{z}^2$$

and its rotations show that the bounds are sharp.

The following covering result follows from the left side inequality in Corollary 2.9.

Corollary 2.10. *Let* $f \in TH^0(\phi, \lambda, \sigma, \alpha)$, *the*

$$\left\{ w : |w| < 1 - \frac{1-\alpha}{(2-\alpha(1+\lambda))\phi_2} - \left(1 - \frac{1+\alpha(1-\lambda)}{(2+\alpha(1-3\lambda))\,\phi_2}\right) |b_1| \right\} \subset f(U).$$

Theorem 2.11. *Let* $h_1(z) = z, h_n(z) = z - \frac{(1-\alpha)}{(n(1-\alpha\lambda)-\alpha(1-\lambda))\phi_n} z^n$, *and* $g_n(z) = z + \frac{(1-\alpha)}{\sigma(n(1-\alpha\lambda)+\alpha(1-\lambda))\phi_n} \overline{z}^n$ $(n \geq 1)$. *A function* $f \in TH^0(\phi, \lambda, \sigma, \alpha)$ *if and only if* f *can expressed in the form* $f(z) = \sum_{n=1}^{\infty} (\lambda_n h_n + \gamma_n g_n)$, *where* $\lambda_n \geq 0, \gamma_n \geq 0, \lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n - \sum_{n=1}^{\infty} \gamma_n$.

In particular, the extreme points of the class $TH^0(\phi, \lambda, \sigma, \alpha)$ *are* $\{h_n\}$ *and* $\{g_n\}$, *respectively.*

Proof. Let

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} (\lambda_n h_n + \gamma_n g_n) \\ &= z - \sum_{n=2}^{\infty} \lambda_n \frac{(1-\alpha)}{(n(1-\alpha\lambda)-\alpha(1-\lambda))\phi_n} z^n + \sum_{n=1}^{\infty} \gamma_n \frac{(1-\alpha)}{(n(1-\alpha\lambda)+\alpha(1-\lambda))\phi_n} \overline{z}^n. \end{aligned}$$

Since

$$\sum_{n=2}^{\infty}\frac{n(1-\alpha\lambda)-\alpha(1-\lambda)}{(1-\alpha)}\lambda_n\frac{(1-\alpha)}{(n(1-\alpha\lambda)-\alpha(1-\lambda)\,\phi_n}\phi_n$$
$$+\sigma\sum_{n=1}^{\infty}\frac{n(1-\alpha\lambda)+\alpha(1-\lambda)}{(1-\alpha)}\gamma_n\frac{(1-\alpha)}{\sigma\,(n(1-\alpha\lambda)+\alpha(1-\lambda)\,\phi_n}\phi_n$$
$$=\sum_{n=2}^{\infty}\lambda_n+\sum_{n=1}^{\infty}\gamma_n=1-\lambda_1\leq 1,$$

it follows from Theorem 2.7 that $f\in TH^0\left(\phi,\lambda,\sigma,\alpha\right)$.

Conversely, if $f\in TH^0\left(\phi,\lambda,\sigma,\alpha\right)$, then $a_n\leq\frac{(1-\alpha)}{(n(1-\alpha\lambda)-\alpha(1-\lambda))\phi_n}$ and $b_n\leq\frac{(1-\alpha)}{(n(1-\alpha\lambda)+\alpha(1-\lambda))\phi_n}$.

Set $\lambda_n=\frac{n(1-\alpha\lambda)-\alpha(1-\lambda)}{(1-\alpha)}a_n\phi_n$, $\gamma_n=\frac{(n(1-\alpha\lambda)+\alpha(1-\lambda))\sigma}{(1-\alpha)}b_n\phi_n$, $\lambda_1=1-\sum\limits_{n=2}^{\infty}\lambda_n-\sum\limits_{n=1}^{\infty}\gamma_n$, then

$$\sum_{n=1}^{\infty}\left(\lambda_n h_n+\gamma_n g_n\right)=z-\sum_{n=2}^{\infty}a_n z^n+\sum_{n=1}^{\infty}b_n\bar{z}^n=f(z).$$

This proves Theorem 2.11. □

3. Applications

We can derive new results by taking $\phi(z)$ as follows:

1- $\phi(z)=z+\sum\limits_{n=2}^{\infty}\left[\frac{1+\ell+\gamma(n-1)}{1+\ell}\right]^m z^n$ (see [4, 7] and [15] with $p=1$), where $\ell>-1,\gamma\geq 0$ and $m\in\mathbb{Z}=\{0,\pm1,\pm2,...\}$.

2- $\phi(z)=z+\sum\limits_{n=2}^{\infty}\Gamma_{n-1}(\alpha_1)z^n$ (see [6]), where $\Gamma_{n-1}(\alpha_1)$ is given by (2.5).

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References

[1] O. P. Ahuja, J. M. Jahangiri and H. Silverman, Convolutions for special classes of harmonic univalent functions, *Appl. Math. Lett.* **16** (2003), no. 6, 905–909.

[2] H. A. Al-kharsani and R. A. Al-Khal, Univalent harmonic functions, *Inequal. Pure Appl. Math.* **8** (2007), no. 2, 8 pages.

[3] R. M. Ali, B. A. Stephen and K. G. Subramanian, Subclasses of harmonic mapping defined by convolution, *Appl. Math. Letters* **23** (2010), no. 10, 1243–1247.

[4] A. Catas, On certain classes of p-valent functions defined by multiplier transformations, in Proceedings of the International Symposium on Geometric Function Theory and Applications: GFTA 2007 Proceedings (İstanbul, Turkey; 20-24 August 2007) (S. Owa

and Y. Polatoglu, Editors), pp. 241–250, TC İstanbul Kűltűr University Publications, Vol. 91, TC İstanbul Kűltűr University, İstanbul, Turkey, 2008.

[5] J. Clunie and T. Sheil-Small, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. A. I. Math.* **9** (1984) 3–25.

[6] J. Dziok and H. M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric function, *Appl. Math. Comput.* **103** (1999), no. 1, 1–13.

[7] R. M. El-Ashwah and M. K. Aouf, Some properties of new integral operator, *Acta Univ. Apulensis Math. Inform.* **24** (2010), 51–61.

[8] F. M. Sakar and H. Guney, On harmonic uniformly starlike functions defined by an integral operator, Acta Univ. Apulensis, **28** (2011) 293–301.

[9] J. M. Jahangiri, Coefficient bounds and univalence criteria for harmonic functions with negative coefficients, *Ann. Univ. Marie-Curie Sklodowska Sect. A* **52** (1998), no. 2, 57–66.

[10] J. M. Jahangiri, Harmonic functions starlike in the unit disk, *J. Math. Anal. Appl.* **235** (1999), no. 2, 470–477.

[11] J. M. Jahangiri, G. Murugusungdaramoorthy and K. Vijaya, Salagean-type harmonic univalent functions, *Southwest J. Pure Appl. Math.* **2** (2002), no. 2, 77–82.

[12] J. M. Jahangiri, G. Murugusungdaramoorthy and K. Vijaya, Starlikeness of harmonic functions defined by Ruscheweyh derivatives, *J. Indian Acad. Math.* **26** (2004), no. 1, 191–200.

[13] G. Murugusundaramoorthy, K. vijaya and B. A. Frasin, A subclass of harmonic functions with negative coefficients defined by Dziok-Srivastava operator, *Tamkang J. Math.* **42** (2011), no. 4, 463–473.

[14] M. Ozturk, S. Yalcin and M. Yamankaradenz, Convex subclass of harmonic starlike functions, *Appl. Math. Comput.* **154** (2004), no. 2, 449-459.

[15] J. K. Prajapat, Subordination and superordination preserving properties for generalized multiplier transformation operator, *Math. Comput. Modelling* **55** (2012), no. 3-4, 1456–1466.

[16] T. Rosy, B. Adolf Stephen, K. G. Subramanian and J. M. Jahangiri, Goodman-Ronning-type harmonic univalent functions, *Kyungpook Math. J.* **41** (2001), no. 1, 45–54.

[17] B. Saker and S. S. Eker, On Salagean-type harmonic functions, *General Math.* **15** (2007), no. 2-3, 52-63.

[18] T. Sheil-Small, Constants for planer harmonic mappings, *J. London Math. Soc. (2)* **42** (1990), no. 2, 237–248.

[19] H. Silverman, Harmonic univalent functions with negative coefficients, *J. Math. Anal. Appl.* **220** (1998), no. 1, 283–289.

[20] H. Silverman and E. M. Silvia, Subclasses of harmonic univalent functions, *New Zealand J. Math.* **28** (1999), no. 2, 275–284.

[21] T. V. Sudharsan, R. Ezhilarasi, B. A. Stephen and K. G. Subramnian, A new subclass of Salagean-type harmonic univalent functions, *Chamchuri J. Math.* **2** (2010), no. 1, 81–92.

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