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# Modulational instability of a weakly relativistic ion acoustic wave in a warm plasma with nonthermal electrons

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An investigation has been made of modulational instability of a nonlinear ion acoustic wave in a weakly relativistic warm unmagnetized nonthermal plasma whose constituents are an inertial ion fluid and nonthermally distributed electrons. Up to the second order of the perturbation theory, a nonlinear Schrödinger type (NST) equation for the complex amplitude of the perturbed ion density is obtained. The coefficients of this equation show that the relativistic effect, the finite ion temperature and the nonthermal electrons modify the condition of the modulational stability. The association between the small-wavenumber limit of the NST equation and the oscillatory solution of the Korteweg–de Varies equation, obtained by a reductive perturbation theory, is satisfied.

**Keywords:** derivative expansion method, warm plasma, ion acoustic waves, modulational instability
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## 1. Introduction

The modulation of one-dimensional (1D) ion acoustic waves (IAWs) in an unmagnetized, collision-free plasma in the finite-wavenumber region has received a great deal of attention.[1−4] These studies have shown that the amplitude of the perturbed ion density is governed by a nonlinear Schrödinger type (NST) equation. However, these studies were concerned with non-relativistic cold plasmas.[5−7] For an envelope soliton, there has been a great deal of interest in studying the modulational instability of different wave modes in plasma, because of its importance in stable wave propagation. Experimental observations of the modulational instability of the monochromatic IAW have been reported by Watanabe.[8] On the other hand, most of the investigations of the propagation of the nonlinear ion acoustic solitons have been developed on the basis of the reductive perturbation theory. In this theory a set of coupled nonlinear partial differential equations describing the system is reduced to the Korteweg–de Varies (KdV) equation or to the NST equation, depending on whether the system is a weakly dispersive or strongly dispersive medium.

Recently,using the reductive perturbation technique, Xue *et al*[9] study the modulational instability of the modulated IAW in a warm plasma. However, this problem had been investigated by El-Labany[4] and El-Labany and El-Hanbaly[10]using the derivative expansion method by Kawahara.[11]

A few years ago, motivated by the observations of solitary structures with density depletions. Cairns and co-workers[12−14] have considered a plasma consisting of nonthermal electrons, with excess of energetic particles and cold ions and have shown that it is possible to obtain both positive (compressive) and negative (rarefactive) solitary waves.

In this paper, we study the modulational instability of the modulated weakly relativistic IAW in a collisionless unmagnetized warm plasma with nonthermal electrons, using the derivative expansion technique.It is organized as follows. In section 2 a weakly relativistic NST equation is derived from the fluid equations describing the system. In section 3 the stability analysis of this equation is investigated. In section 4 the relationship between weakly relativistic ion modulation modes in the small wavenumber region and in the finite wavenumber region is investigated. In section 5

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a concluding discussion is presented and a comparison with previous results is considered.

## 2. Basic equations and derivation of the NST equation

Consider a simple weakly relativistic plasma model that includes one warm ion species together with nonthermal electrons. Also, we assume that the plasma is unmagnetized, collisionless and ionization-free. The one-dimensional basic equations can be written in a non-dimensional form

$$\frac{\partial n}{\partial t}+\frac{\partial\,(nu)}{\partial\,x}=0, \tag{1}$$

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)(\gamma u)+\frac{\sigma}{n}\frac{\partial p}{\partial x}+\frac{\partial\phi}{\partial x}=0, \tag{2}$$

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)p+3p\frac{\partial(\gamma u)}{\partial x}=0, \tag{3}$$

$$\frac{\partial^2\phi}{\partial x^2}=n_{\rm e}-n, \tag{4}$$

$$n_{\rm e}={\rm e}^{\phi}[1-(1-\nu)(1-\phi)\phi], \tag{5}$$

where $n$ and $n_{\rm e}$ are the densities of the ions and electrons respectively, $u$ is the flow velocity of the ions, $p$ is the pressure, $\phi$ is the electrostatic potential, $\nu=\dfrac{(1-\alpha)}{(1+3\alpha)}$ determines the population of nonthermal electrons in the present plasma model, $x$ is the space coordinate, $t$ is the time variable and $\sigma\ll 1$ is the ratio of ion temperature $T_{\rm i}$ to electron temperature $T_{\rm e}$. The relativistic factor is approximated by its expansion up to the second term because of a weakly relativistic effect,

$$\gamma\approx\left(1+\frac{u^2}{2C^2}\right), \tag{6}$$

where $C$ is the velocity of light.

All physical quantities in Eqs.(1–5) as well as the space and time variables have been rendered dimensionless in terms of the following quantities: ion thermal velocity $=(k_{\rm B}T_{\rm e}/m)^{1/2}$, ion pressure $=n_0k_{\rm B}T_i$, the Debye length $\lambda_{\rm D}=(k_{\rm B}T_{\rm e}/4\pi\,e^2n_0)^{1/2}$, a characteristic potential $(k_{\rm B}T_{\rm e}/e)$ and a characteristic time $(4\pi e^2n_0/m)^{-1/2}$ (the inverse of the ion plasma frequency; $\omega_{pi}$=1), where $k_{\rm B}$ is Boltzmann's constant and $m$ is the ion mass.

According to the general method of the derivative expansion,[4,11] we introduce the stretched variables

$$\tau_i=\varepsilon^i t, \tag{7a}$$

$$\xi_i=\varepsilon^i(x-\lambda\,t);\ \ (i=1,2,\ldots),\xi_0=x \tag{7b}$$

where the parameter $\lambda$ is the group velocity, to be determined later; and the smallness parameter $\varepsilon$ represents the size of the perturbed amplitude.

To derive the NST equation describing the propagation of the nonlinear ion acoustic waves from the basic equations (1–5), we expand all the quantities asymptotically in the smallness parameter $\varepsilon$ about their equilibrium values

$$G(x,t)=G_{\rm o}+\sum_{m-1}^{\infty}\varepsilon^m\times\sum_{L=-\infty}^{\infty}G_m^{(L)}(\tau_1\,,\tau_2,\ldots,\xi_1,\,\xi_2,\ldots), \tag{8}$$

where

$$G_m^{(L)}=[n_m^{(L)}u_m^{(L)}p_m^{(L)}\phi_m^{(L)}],$$

$$G_{\rm o}=[1\ \ u_0\,1\,0]\quad\text{and}\quad\theta=kx-\omega t.$$

Here $G_m^{(L)}$ satisfies the reality condition $G_m^{(L)}=G_m^{*(-L)}$, where the asterisk denotes the complex conjugate. The dependence on the scales $\tau_1,\tau_2,\ldots,\,\xi_1,\xi_2,\ldots$ is chosen in such a way that the secular terms are eliminated.

Substituting Eqs.(7) into the basic equations (1–5), we obtain, to first order in $\varepsilon$ and $L=1$ components

$$u_1^{(1)}=\frac{\hat{w}}{k}n_1^{(1)},\,p_1^{(1)}=3\gamma_1n_1^{(1)}\ \text{ and}\,\phi_1^{(1)}=\frac{n_1^{(1)}}{k^2+\nu}, \tag{9}$$

where $\gamma_1=1+\dfrac{3\,u_0^2}{2\,C^2},\hat{w}=\omega-ku_0$and the linear dispersion and the group velocity $\lambda$ are given respectively by

$$\gamma_1\hat{w}=3\gamma_1k^2\sigma+\frac{k^2}{k^2+\nu}, \tag{10a}$$

$$\lambda=u_0+\frac{k}{\hat{w}\gamma_1}[3\gamma_1\sigma+\frac{\nu}{(k^2+\nu)^2}]. \tag{10b}$$

The $L=0$ components of $O\,(\varepsilon)$ give

$$n_1^{(0)}=n_{\rm e1}^{(0)}\ \text{and}\quad\phi_1^{(0)}=\frac{n_1^{(0)}}{\nu}.$$

Turning to $O\,(\varepsilon^2)$of the reduced equations, we can obtain, for $L=0$ components,

$$\frac{\partial\,n_1^{(0)}}{\partial\,\xi_1}=\frac{\partial\,u_1^{(0)}}{\partial\,\xi_1}=\frac{\partial\,p_1^{(0)}}{\partial\,\xi_1}=\frac{\partial\,\phi_1^{(0)}}{\partial\,\xi_1}=0,$$

$$\phi_1^{(0)}=0\ \text{ and }\ \phi_2^{(0)}=\frac{[n_2^{(0)}-\phi_1^{*(1)}\phi_1^{(1)}]}{\nu};$$

for $L=1$ components,

$$\frac{\partial n_1^{(1)}}{\partial\tau_1}=0,\ u_2^{(1)}=\left(\frac{\hat{w}}{k}\right)n_2^{(1)}+\frac{\rm i}{k}(\frac{\hat{w}}{k}-\lambda)\frac{\partial n_1^{(1)}}{\partial\xi_1},$$

$p_2^{(1)}=3\gamma_1 n_2^{(1)}$ and $\phi_2^{(1)}=\dfrac{n_2^{(1)}}{(k^2+\nu)}+\dfrac{2\mathrm{i}k}{(k^2+\nu)^2}\dfrac{\partial n_1^{(1)}}{\partial\xi_1}$;

for $L=2$ components,

$$n_2^{(2)}=(1+A_1)n_1^{(1)^2},\; u_2^{(2)}=\frac{\hat{w}}{k}A_1 n_1^{(1)^2},$$

$$p_2^{(2)}=\frac{3\gamma_1}{2}(2A_1+B_1)n_1^{(1)^2}$$

and

$$\phi_2^{(2)}=\frac{(2A_1-A_2)}{2(k^2+\nu)}n_1^{(1)^2},$$

where

$$A_1=1+\frac{(3\nu^2-1)}{6k^2\nu}+\frac{\gamma_1\hat{w}^2}{6k^2\nu}\left(1+\frac{\nu}{4k^2}\right)+\frac{\gamma_1\sigma H}{2},$$

$$A_2=2-\nu-\left(\frac{2\gamma_2}{\gamma_1}\right)\frac{\hat{w}}{k}+3\gamma_1\sigma(1+3\gamma_1)(k^2+\nu),$$

$$H=\frac{-1}{\nu}+\left[\frac{(1+3\gamma_1)(k^2+\nu)(4k^2+\nu)}{k^2}\right],$$

$$B_1=1+3\gamma_1+\left(\frac{2\gamma_2}{\gamma_1}\right)\frac{\hat{w}}{k}\quad\text{and}\quad\gamma_2=\frac{3u_0}{2C^2}.$$

Equating terms of $O(\varepsilon^3)$ in the set of equations (1–5), we have for the $L=0$ components,

$$u_2^{(0)}=\left(\frac{|\,n_1^{(1)}|^2-C_1}{Z}\right)\left[\frac{2\hat{w}}{k}\left(\frac{1}{\nu}+\frac{3\gamma_1\sigma B_1}{2}\right)+B_2\right],$$

$$n_2^{(0)}=\left(\frac{|\,n_1^{(1)}|^2-C_1}{Z\,\tilde{\lambda}}\right)\left[\frac{2\hat{w}}{k}\left(\frac{1}{\nu}+\frac{3\gamma_1\sigma B_1}{2}+Z\right)+B_2\right],$$

$$p_2^{(0)}=3\gamma_1\tilde{\lambda}\left(\frac{|\,n_1^{(1)}|^2-C_1}{Z}\right)\times\left[\frac{2\hat{w}}{k}\left(\frac{1}{\nu}+\frac{3\gamma_1\sigma B_1}{2}+\frac{B_1 Z}{2}\right)+B_2\right],$$

$$\phi_2^{(0)}=\left(\frac{|\,n_1^{(1)}|^2-C_1}{\nu\,\tilde{\lambda}\,Z}\right)\times\left[\frac{2\hat{w}}{k}\left(\frac{1}{\nu}+\frac{3\gamma_1\sigma B_1}{2}+Z\right)+B_2\right]-\frac{|\,n_1^{(1)}|^2}{\nu\,(k^2+\nu)},$$

where

$$B_2=\frac{\tilde{\lambda}(k^2+\nu-[1/\nu])}{(k^2+\nu)^2}-2\gamma_2\tilde{\lambda}^2\frac{\hat{w}^2}{k^2},$$

$Z=\gamma_1\tilde{\lambda}^2-3\gamma_1\sigma-(1/\nu)$, $\tilde{\lambda}=\lambda-u_0$; and $C_1$ is a constant of integration, independent of $\xi_1$.

Using the first-and second-order solutions for the $L=1$ components of the set produced equations of $O(\varepsilon^3)$, the resonant terms leads to the non-secular condition and the NST equation, as follows,

$$\mathrm{i}\frac{\partial n_1^{(1)}}{\partial\tau_2}+S\frac{\partial^2 n_1^{(1)}}{\partial\xi_1^2}+RC_1\,n_1^{(1)}+Q\,|\,n_1^{(1)}|^2\,n_1^{(1)}=0.\quad(11)$$

The coefficients of the linear, dispersive and nonlinear terms are given respectively by

$$R=\frac{k}{2\gamma_1\tilde{\lambda}}\left[\frac{2(k^2+\nu-[1/\nu])}{(k^2+\nu)^2}-3\gamma_1\sigma(2-3\gamma_1 B_1)\right]+\frac{k^2C_2}{2\gamma_{1\hat{w}Z}}\left[\frac{2\gamma_1\hat{w}}{k}-\frac{2\gamma_2}{\gamma_1(k^2+\nu)}+\left(\frac{(k^2+\nu-[1/\nu])}{\tilde{\lambda}(k^2+\nu)^2}+\frac{3\gamma_1\sigma(3\gamma_1-1)}{\tilde{\lambda}}\right)\right],$$

$$S=\frac{-k^2}{2\hat{w}\gamma_1(k^3+\nu)^3}\left[4\nu-\frac{\tilde{\lambda}k(k^2+\nu)}{\hat{w}}\right]=\frac{1}{2}\frac{\mathrm{d}\lambda}{\mathrm{d}k},$$

$$Q=-R-Q_1,$$

$$Q_1=\frac{k^2}{2\gamma_{1\hat{w}}}\left\{\begin{array}{c}\dfrac{-3\sigma\left[-\gamma_1\dfrac{4k+6A_1\gamma_1k+4A_1\gamma_2w-3\gamma_1B_1k}{2k}k^3+6\gamma_2w^3\right]}{k^3}\\[2ex] -\gamma_1A_1w^2\dfrac{-1+2\gamma_2\dfrac{w}{\gamma_1k}}{k^2}-2\gamma_2w^3\dfrac{-1+\dfrac{3}{4}\dfrac{w}{\gamma_2kC^2}}{k^3}\\[2ex] +9\gamma_1\sigma\dfrac{\hat{w}^2}{k^2}+\dfrac{2A_1+1}{k^2+\nu}-\dfrac{1}{2}\dfrac{-A_2+2A_1-\dfrac{3\nu-4+\dfrac{2}{\nu}}{k^2+\nu}}{(k^2+\nu)^3}\end{array}\right\},$$

and

$$C_2=B_2+\frac{2\hat{w}}{\nu\,k}\left(1+\frac{3}{2}\gamma_1\sigma\nu B_1\right).$$

(see Appendix A).

Equation (11) governs the evolution of the complex amplitude of the nonlinear IAW in the finite-wavenumber region, propagating in a weakly relativistic warm plasma with nonthermal electrons.

## 3. Stability analysis

The sign of the dispersion coefficient $Q$ characterizes the amplitude of the modulated IAWs defined by the NST Eq. (11). The stability criterion of this equation exhibits a modulational stability of the amplitude of the wave envelope if $SQ < 0$, and exhibits a modulational instability if $SQ > 0$. We know that the former case gives rise to a dark envelope soliton and the latter gives a bright envelope soliton. However, the expressions of $S$ and $Q$ show that the stable and unstable regions are modified due to the finite ion temperature ($\sigma$), relativistic effect ($\gamma_1$, $\gamma_2$)and nonthermal electron ($\nu$). Since $S$ is always negative for $w > ku_0$, one has to determine the value of the critical wavenumber $k_{\rm C}(\sigma, \gamma_1, \nu)$at which $Q$ changes its sign. Then, for all values of $k < k_{\rm C}$, the wave has a modulational stability, while the modulational instability occurs in the region $k > k_{\rm C}$. Applying the dispersion relation and after straightforward manipulation, we can reduce the nonlinear coefficient in the form:

$$Q(\sigma, \gamma_1, \nu) = \frac{\sum_{i=1}^{10} w_i k^{2(10-i)}}{12\gamma_1 \hat{w}\nu D(k^{2+\nu})^4}, \qquad (12)$$

where

$$D = 3(k^2+\nu)(1+\gamma_1\sigma k^2) + (k^4/\nu),$$

and $w_i\,(i = 1, 2, \ldots, 10)$ is the weight of each term.

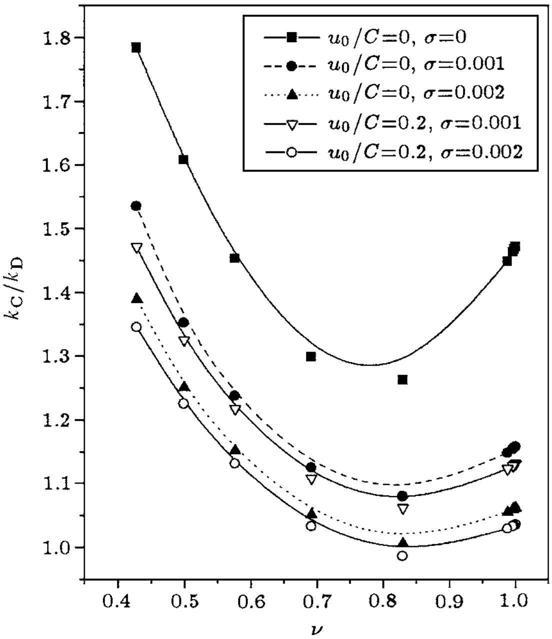


**Fig.1.** The dependence of $k_{\rm C}$ (in units of $k_{\rm D}$) on the nonthermal parameter $\nu$ for certain values of relativistic parameter $u_0/C = 0$ (non-relativistic) and 0.2. $k_{\rm C}$ first decreases rapidly then increases slowly for higher values of $\nu$,as it approaches the isothermal value ($\nu = 1$).

Figure 1 shows the dependence of $k_{\rm C}$, in units of $k_{\rm D}$, on the nonthermal parameter $\nu$. It is shown that $k_{\rm C}$ first decreases rapidly, then increases slowly for the higher values of $\nu$. It is proved that as the ion temperature $\sigma$ or the relativistic effect $(u_0/C)$ increase, $k_{\rm C}$ decreases; but the variation with the relativistic effect is small.

## 4. Oscillatory solution of the KdV equation

In the small-wavenumber region, Eq. (11) reduces to

$$\begin{aligned} &\mathrm{i}\frac{\partial n_1^{(1)}}{\partial \tau_2} - \frac{3\,k\,b}{2}\frac{\partial^2 n_1^{(1)}}{\partial \xi_1^2} - \frac{3\,k\,bC_1}{2} n_1^{(1)} \\ &+ \frac{a^2 \mid n_1^{(1)}|^2 n_1^{(1)}}{3\,k\,\nu^2\,b} = 0, \end{aligned} \qquad (13)$$

where

$$\begin{aligned} a =& \left\{ \frac{-1 + 3[1+\gamma_1\sigma\nu + 3\gamma_1^2\sigma\nu]\nu^2}{2\nu^2\sqrt{\gamma_1([1/\nu] + 3\gamma_1\sigma)}} - \frac{\gamma_2}{\gamma_1^2} \right\}, \\ b =& [\nu^2\sqrt{\gamma_1(1+3\gamma_1\sigma\nu)}]^{-1}. \end{aligned}$$

(see Appendix A).

Applying the reductive perturbation theory, the complex amplitude of the perturbed density in a weakly relativistic warm plasma with nonthermal electrons in the small-wavenumber limit is governed by the KdV equation, which is given by:

$$\frac{\partial \tilde{n}}{\partial \tau} + \frac{a}{\nu}\tilde{n}\frac{\partial \tilde{n}}{\partial \xi} + \frac{b}{2}\frac{\partial^3 \tilde{n}}{\partial \xi^3} = 0, \qquad (14)$$

where $\tilde{n}$ is the perturbed density, and $\tau$ and $\xi$ are given by

$$\xi = \mu^{\frac{1}{2}}(x - \bar{\lambda}t), \tau = \mu^{\frac{3}{2}}t.$$

Here $\mu$ is the ordering parameter and the wavenumber $k = O\,(\mu^{\frac{1}{2}})$.[4]

To obtain an oscillatory solution of the KdV Eq. (14), we follow El-Labany[4] and El-Labany and El-Hanbaly,[10] expanding ($\tilde{n}$ ) as

$$\tilde{n} = \sum_{m=1}^{\infty} \varepsilon^m \sum_{L=-\infty}^{\infty} \tilde{n}_m^L(\rho, \zeta)\mathrm{exp.}[\mathrm{i}L(k\xi - \delta\tau)], \quad (15)$$

where the stretched variables $\rho$ and $\zeta$ are related to $\xi$ and $\tau$ by

$$\rho = \varepsilon(\xi - \beta\tau), \qquad \zeta = \varepsilon^2\tau. \qquad (16)$$

The parameters $\delta$ and $\beta$ will be determined later. Substituting Eqs.(15) and (16) into Eq.(14), we obtain the following reduced equation of order $m$:

$$\begin{aligned}&-\mathrm{i}\delta L\tilde{n}_m^{(L)}-\beta\frac{\partial\tilde{n}_{m-1}^{(L)}}{\partial\rho}+\frac{\partial\tilde{n}_{m-2}^{(L)}}{\partial\zeta}-\frac{\mathrm{i}}{2}b(kL)^3\tilde{n}_m^{(L)}\\&-\frac{3}{2}b(kL)^2\frac{\partial\,\tilde{n}_{m-1}^{(L)}}{\partial\rho}+\frac{3}{2}\mathrm{i}b(kL)\frac{\partial^2\tilde{n}_{m-2}^{(L)}}{\partial\,\rho^2}+\frac{b}{2}\frac{\partial^3\,\tilde{n}_{m-3}^{(L)}}{\partial\rho^3}\\&+\frac{\mathrm{i}ak}{\nu}\sum_{m'=1}^{m-1}\sum_{L=-\infty}^{\infty}(L-L')\tilde{n}_{m'}^{(L)}\tilde{n}_{m-m'}^{(L-L')}\\&+\frac{a}{\nu}\sum_{m=1}^{m-2}\sum_{L'=-\infty}^{\infty}\tilde{n}_{m'}^{(L')}\frac{\partial\tilde{n}_{m-m'-1}^{(L-L')}}{\partial\rho}=0.\end{aligned}\tag{17}$$

The first-order terms ($m=1$) with $L=\pm1$ lead to the linear dispersion relation

$$\delta=\frac{-1}{2}bk^3.$$

The second-harmonic components of the second-order terms of the reduced Eq. (17) gives

$$\tilde{n}_2^{(2)}=\frac{a}{3\nu bk^2}\tilde{n}_1^{(1)^2};$$

while the components $L=\pm1$ of this order lead to the compatibility endurance condition for non-trivial solution, i.e.

$$\beta=\frac{-3}{2}b\,k^2.$$

The zeroth-harmonic components of the third-order terms of the $m$th reduced equation are determined as

$$\tilde{n}_2^{(0)}=\frac{-2\,a}{3\,\beta\,k^2}(|\,n_1^{(1)}|^2-C_1),$$

where $C_1$ is a constant independent of $\rho$.

Finally, the $L=1$ terms of the third-order reduced equation give the NST equation (Eq.(13)). In terms of the solution of this equation, the oscillatory solution of Eq.(14) is expressed as

$$\begin{aligned}\tilde{n}=&\varepsilon\tilde{n}_1^{(1)}\Bigg\{\Bigg(\Bigg[x-(u_0+\sqrt{\frac{1+3\gamma_1\sigma\nu}{\nu\gamma_1}}\\&-\frac{3k^2}{\sqrt{\nu\gamma_1[1+3\gamma_1\sigma\nu]}}\Bigg)t\Bigg],\varepsilon t\Bigg\}\\&\times\exp\Bigg\{\mathrm{i}k\Bigg[x-\Bigg(u_0+\sqrt{\frac{1+3\gamma_1\sigma\nu}{\nu\gamma_1}}\\&-\frac{k^2}{\sqrt{\nu\gamma_1[1+3\gamma_1\sigma\nu]}}\Bigg)t\Bigg]\Bigg\}.\end{aligned}$$

## 5. Conclusions

To verify our results, we can consider a special case of a weakly relativistic warm plasma with isothermal electrons and then make comparisons with published works. For $\nu=1$(isothermal electrons), all the coefficients in this work agree exactly with those obtained by El-Labany.[4] Also, if we neglect the relativistic effect (put $\gamma_1=1$, $\gamma_2=0$), we can acquire the result of Xue Ju-Kui *et al* Thus, this paper can be considered as the generalization of the work done by El-Labany[4]with the inclusion of nonthermal electrons.

We conclude that, on the basis of the derivative expansion method, the modulation of nonlinear IAW in a weakly relativistic warm plasma with nonthermal electrons has been investigated. A NST equation describing the evolution of the complex amplitude of the perturbed ion density in the finite wavenumber region is obtained. The coefficients of this equation have been shown to be strongly dependent on the ion temperature $\sigma$, the ion streaming velocity ($\gamma_1$, $\gamma_2$) and the nonthermal parameter $\nu$. We can summarize these effects as follows:

1. The critical wavenumber $k_{\mathrm{C}}$ decreases as $\sigma$ increases for fixed values of ion streaming, and also $k_{\mathrm{C}}$ decreases as the relativistic effect increases but its variation with the latter is slightly slow.

2. The $k_{\mathrm{C}}$ first decreases rapidly as $\nu$ increases, then it increases slowly as $\nu$ increases. This means there exists a critical nonthermal population of electrons to sustain the system in the stable state.

Moreover, we have derived the relation between the weakly relativistic ion modulation modes described by the KdV equation in the small wavenumber region and the NST equation in the finite wavenumber region. The dispersion and the nonlinear coefficients of the KdV equation are exactly the same as those of the small wavenumber limit of the NST equation derived in the finite wavenumber region. In addition, the oscillatory solution of the KdV equation satisfies the NST equation.

# Appendix A

To evaluate the coefficients of the NST equation for small $k$, we first calculate the different terms appearing in these coefficients.

From Eqs.(10), as $k\to0$, we have

$$\left(\frac{\hat{w}}{k}\right)\approx\tilde{\lambda}\approx\left(\frac{1+3\gamma_1\sigma\nu}{\nu\gamma_1}\right)^{\frac{1}{2}},$$

thus

$$\begin{aligned}\left[\frac{1}{Z}\right]=&\Bigg[\frac{\nu(k^2+\nu)}{k^2}(k^2\nu\{3\gamma_1\sigma(k^2+\nu)+1\}\\&-(k^2+3\nu)(k^2+\nu))^{-1}\Bigg]\\\approx&\frac{-\nu^2}{3k^2},\end{aligned}$$

where the square-bracket notation indicates that the quantity is evaluated at $k \to 0$. Also

$$[A_1] = \left[ \begin{array}{c} 1 + \frac{3\nu^2 - 1}{6k^2\nu} + \frac{\gamma_1}{6\nu}(\frac{\hat{w}}{k})^2 - \frac{4\gamma_2}{3\gamma_1}(\frac{\hat{w}}{k})(1 + \frac{\nu}{4k^2}) \\ + \frac{\gamma_1 \sigma}{2\,\nu\, k^2}\{-k^2 + \nu(1+3\gamma_1)(k^2+\nu)(4k^2+\nu)\} \end{array} \right]$$

$$\approx 1 + \frac{3\nu^2 - 1}{6k^2\nu} + \frac{(1+3\gamma_1\sigma\nu)}{6\nu^2} - \frac{4\gamma_2}{3\gamma_1^{3/2}}\left(\frac{1+3\gamma_1\sigma\nu}{\nu}\right)^{1/2} \times (1 + \frac{\nu}{4k^2}) + \frac{\gamma_1\sigma}{2\nu\, k^2}\{-k^2 + (1+3\gamma_1)\nu(5k^2+\nu)\},$$

$$[A_2] = \left[2 - \nu - \frac{2\gamma_2}{\gamma_1}\left(\frac{\hat{w}}{k}\right) + 3\gamma_1\sigma(k^2+\nu)\right] \approx 2 - \nu - \frac{2\gamma_2}{\gamma_1^{3/2}}\left(\frac{1+3\gamma_1\sigma\nu}{\nu}\right)^{1/2} + 3\gamma_1\sigma\nu(1+3\gamma_1),$$

$$[B_1] = \left[1 + 3\gamma_1 + \frac{2\gamma_2}{\gamma_1}\left(\frac{\hat{w}}{k}\right)\right] \approx 1 + 3\gamma_1 + \frac{(1+3\gamma_1\sigma\nu)^{\frac{1}{2}}}{\nu},$$

$$[B_2] = \left[\frac{\tilde{\lambda}(k^2+\nu-\frac{1}{\nu})}{(k^2+\nu)^2} - 2\gamma_2\tilde{\lambda}\left(\frac{\hat{w}}{k}\right)^2\right] \approx \frac{\tilde{\lambda}}{\nu^3}(\nu^2-1) - 2\gamma_2\left(\frac{1+3\gamma_1\sigma\nu}{\gamma_1\nu_1}\right)^2,$$

and

$$[C_2] = \left[\frac{2\hat{w}}{k}\left(\frac{1}{\nu} + \frac{3\gamma_1\sigma B_1}{2}\right) + B_2\right] \approx 2\left(\frac{1+3\gamma_1\sigma\nu}{\nu\gamma_1}\right)^{1/2}\left\{\frac{1}{\nu} + \frac{3\gamma_1\sigma}{2}\left(1 + 3\gamma_1 + \frac{2\gamma_2}{\gamma_1^{3/2}}\left(\frac{1+3\gamma_1\sigma\nu}{\nu}\right)^{1/2}\right)\right\} + \frac{\tilde{\lambda}}{\nu^3}(\nu^2-1) - 2\gamma_2\left(\frac{1+3\gamma_1\sigma\nu}{\gamma_1\nu_1}\right)^2.$$

Thus the coefficients $S$, $R$ and $Q$ are given by

$$[S] = -\left[\frac{k^2}{2\gamma_1\hat{w}(k^2+\nu)^3}\left(4\nu - \frac{\tilde{\lambda}k}{\hat{w}}(k^2+\nu)\right)\right] \approx -\frac{3k}{2\nu^2}\left\{\gamma_1\left(\frac{1}{\nu} + 3\gamma_1\sigma\right)\right\}^{-1/2} = \frac{-3}{2}kb,$$

$$[R] = \frac{k^2 C_2}{2\gamma_2 Z\hat{w}}\left\{\frac{2\,\gamma_1\,\hat{w}}{k} - \frac{2\gamma_2}{\gamma_1\nu} + \frac{1}{\tilde{\lambda}}\left\{\frac{\nu^2-1}{\nu^3} + 3\gamma_1\sigma(3\gamma_1 - 1)\right\}\right\}$$

$$\approx \left\{\frac{3\nu^2-1}{2\nu^2} + \frac{3\gamma_1\sigma\nu}{2}(1+3\gamma_1) - \frac{\gamma_2}{\gamma_1^{3/2}}\left(3\gamma_1\sigma + \frac{1}{\nu}\right)^{1/2}\right\} \times \left(\frac{-2}{3k}\right)\left\{\gamma_1\left(\frac{1}{\nu} + 3\gamma_1\sigma\right)\right\}^{1/2},$$

$$[Q_1] = \left(\frac{1}{2}\right)\left\{\gamma_1\left(\frac{1}{\nu} + 3\gamma_1\sigma\right)\right\}^{-1/2} \times \left[kA_1\left\{\frac{2\nu^2-1}{2\nu^2} + \gamma_1\left(\frac{\hat{w}}{k}\right)^2\left(1 - \frac{2\gamma_2}{\gamma_1}\frac{\hat{w}}{k}\right) + 3\gamma_1\sigma\left(3\gamma_1 + \frac{2\,\gamma_2\,\hat{w}}{\gamma_1 k}\right)\right\}^2\right]$$

$$= \left\{\frac{2\nu^2-1}{2\nu^2} + \frac{3\gamma_1\sigma\nu}{2}(1+3\gamma_1) - \frac{\gamma_2}{\gamma_1^{3/2}}\left(3\gamma_1\sigma + \frac{1}{\nu}\right)^{1/2}\right\}^2$$

and $[Q] = -[Q_1] - [R] = [Q_1]$.

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