Dust acoustic solitary waves and double layers in a dusty plasma with two-temperature trapped ions

S. K. El-Labany, W. F. El-Taibany, a) A. A. Mamun, b) and Waleed M. Moslem c)

Physics Department, Faculty of Science-Damietta, Mansoura University, Damietta El-Gedida, P.O. 34517, Egypt

(Received 25 August 2003; accepted 21 November 2003)

The combined effects of trapped ion distribution, two-ion-temperature, dust charge fluctuation, and dust fluid temperature are incorporated in the study of nonlinear dust acoustic waves in an unmagnetized dusty plasma. It is found that, owing to the departure from the Boltzmann ion distribution to the trapped ion distribution, the dynamics of small but finite amplitude dust acoustic waves is governed by a modified Korteweg–de Vries equation. The latter admits a stationary dust acoustic solitary wave solution, which has stronger nonlinearity, smaller amplitude, wider width, and higher propagation velocity than that involving adiabatic ions. The effect of two-ion-temperature is found to provide the possibility for the coexistence of rarefactive and compressive dust acoustic solitary structures and double layers. Although the dust fluid temperature increases the amplitude of the small but finite amplitude solitary waves, the dust charge fluctuation does the opposite effect. The present investigation should help us to understand the salient features of the nonlinear dust acoustic waves that have been observed in a recent numerical simulation study. © 2004 American Institute of Physics. [DOI: 10.1063/1.1643757]

I. INTRODUCTION

There has been a rapidly growing interest in the physics of dusty plasmas not only because dust is an omnipresent ingredient of our universe, but also because of its vital role in understanding different collective processes (mode modification, new eigenmodes, coherent structures, etc.) in astrophysical and space environments.1–6 The consideration of charged dust grains in a plasma does not only modify the existing plasma wave spectra,7,8 but also introduces a number of new novel eigenmodes, such as dust acoustic (DA) waves9,10 dust ion acoustic (DIA) waves,11,12 dust lattice waves,13,14 etc.

Rao et al.9 first reported theoretically the existence of extremely low phase velocity (in comparison with the electron and ion thermal velocities) DA waves where the dust particle mass provides the inertia and the thermal pressures from the electrons and ions give rise to the restoring force. Rao et al.9 have studied the DA solitary waves in an unmagnetized dusty plasma (with cold dust fluid) by using the reductive perturbation method. Motivated by the experimental observation9 of such low phase velocity DA waves, Mamun et al.15 have investigated nonlinear DA waves in a two-component unmagnetized dusty plasma consisting of a negatively charged cold dust fluid and Maxwellian ions.

On the other hand, Roychoudhury and Mukherjee16 showed that the finite dust temperature restricts the region for the existence of nonlinear solitary waves. The dust temperature is important owing to the thermalization with the ions or orbital effects.17 The effects of the dust fluid temperature and nonthermal distribution of ions drastically modify the properties of the large amplitude electrostatic solitary structures.18 Also, El-Labany and El-Taibany19 studied the effects of dust temperature, charge fluctuation and ion streaming on DA waves and double layers (DLs). The effects of finite dust temperature, dust charge fluctuation on DA waves and DLs in a warm dusty plasma system containing trapped and free electrons are investigated.20

Tagare21 extended the model of Mamun et al.15 to a dusty plasma consisting of cold dust particles and two-temperature isothermal ions and studied the existence and properties of DA solitary waves. Using the reductive perturbation theory and pseudopotential method, Xie et al.22 studied the DA solitary waves and DLs in dusty plasma with variable dust charge and two-temperature ions, and they have shown that both compressive and rarefactive solitons as well as DLs exist. Also, the amplitudes of the dust solitary waves become smaller and the regime of Mach number is extended wider for the variable dust charge situation compared to the constant dust charge situation. On the other hand, the topics of nonlinear grain charge variation and electrostatic ion waves23 have been reported by regarding dust grains as point charges, where the Debye length is much larger than the inter-grain distance.

On the other hand, Schekinov24 studied analytically the nonlinear properties of DA waves in a dusty plasma consisting of cold dust grains of constant charge and nonisothermal ions. Mamun25 studied nonlinear small amplitude DA waves considering nonisothermal ions. The effect of nonlinear dust...
grain charging on large amplitude electrostatic waves in a dusty plasma with trapped ions has been studied by Nejoh,26 Kakati and Goswami27 studied nonlinear shock-like DA waves considering nonisothermal ions and adiabatic dust charge variations using the reductive perturbation technique. El-Labany et al.28 revisited the same problem and studied the critical density solitary waves and small amplitude DA waves in a hot dusty plasma with nonisothermal ions. Also, the effect of nonadiabatic dust charge variations on nonlinear DA waves with nonisothermal ions has been investigated by Ghosh et al.29

In most practical dusty plasma experiments, a gas flow is usually introduced, which can be charged quickly, while keeping relatively low temperature. It was found theoretically that two ion acoustic modes can propagate in two-ion plasmas.30 Lakshami and Bharuthram30 studied large amplitude rarefactive DA solitons in a plasma with Boltzmann distributed electrons, ion species at different temperatures and dust grains with constant charges. Roychoudhury and Chatterjee31 studied arbitrary amplitude DLs in dusty plasmas.22 Lakshami and Bharuthram 30 studied large amplitude electrostatic waves in a hot magnetized two-ion-temperature dusty plasma has been studied by Mamun.32 Recent numerical simulation studies33 on linear and nonlinear DA waves exhibit a significant amount of ion trapping in the wave potential. Clearly, there is a departure from the Boltzmann ion distribution and one encounters vortex-like ion distributions in phase space. In this article, we investigate the properties of nonlinear DA waves by incorporating the effects of two-temperature nonisothermal ion distributions that have vortex-like distributions.34–36 As the effects of second component of low-temperature species, dust temperature, and dust charge fluctuation, which have not been considered in these earlier investigations,22,25,28 drastically modify the properties of electrostatic solitary structures,37 in the present work we study the DA solitary structures in a warm unmagnetized dusty plasma which consists of a negatively charged extremely massive dust fluid, isothermal electrons and trapped ions of two different temperatures.

This article is organized as follows: The basic equations governing the dynamics of the nonlinear DA waves are presented in Sec. II. The modified KdV (MKdV) equation and its stationary solitary wave solution are derived in Sec. III. The MKdV equation is then generalized to include the effects of dust charge fluctuation, dust temperature and nonisothermal ion distribution. The formalisms of both rarefactive and compressive solitons as well as DLs are also obtained in the same section (Sec. III). Finally, a brief discussion is presented in Sec. IV.

II. BASIC EQUATIONS

We consider an unmagnetized dusty plasma consisting of massive, negatively charged dust grains, isothermal (Maxwellian) electrons and nonisothermal (trapped or bi-Maxwellian) ions. Thus, at equilibrium we have

\[ n_{ilo} + n_{iho} = n_{eo} + Z_{do} n_{do} , \]

where \( n_{ilo} \) , \( n_{iho} \) , \( n_{eo} \), and \( n_{do} \) are the unperturbed low-temperature ion, high-temperature ion, electron and dust number densities, respectively, and \( Z_{do} \) is the unperturbed number of electrons residing on the dust grain surface. The nonlinear dynamics of one-dimensional low phase speed (in comparison with ion thermal speed) DA waves in such a dusty plasma are described by

\[ \frac{\partial n_d}{\partial t} + \frac{\partial (n_d u_d)}{\partial x} = 0 , \]

\[ \frac{\partial u_d}{\partial t} + u_d \frac{\partial u_d}{\partial x} + 3 \frac{\sigma_d}{m_d} \frac{\partial n_d}{\partial x} - Z_d \frac{\partial \phi}{\partial x} = 0 , \]

\[ \frac{\partial^2 \phi}{\partial x^2} = Z_d n_d + n_e - n_{ilo} - n_{iho} , \]

where \( n_d \) is the dust number density normalized by \( n_{do} \); and \( n_e \) and \( n_{ilo} \) (\( n_{iho} \)) are the electron number density and the low (high) temperature ion number density, respectively, normalized by \( n_{do} Z_{do} \). \( u_d \) is the dust fluid speed normalized by the DA speed \( C_d = (Z_{do} T_{eff}/m_d)^{1/2} \). \( \phi \) is the electrostatic wave potential normalized by \( T_{eff}/e \), \( Z_d \) is the number of electrons residing on the dust grain surface normalized by \( Z_{do} \), \( \sigma_d = (T_d/Z_{do} T_{eff}) \). \( T_{eff} = Z_{do} n_{do} (n_{eo} + (n_{ilo} + n_{iho}))/e \), \( \delta_1 = n_{ilo}/n_{eo} \), \( \delta_2 = n_{iho}/n_{eo} \), and \( m_d \) is the dust particle mass. The space coordinate \( x \) and time \( t \) are normalized by the Debye length \( \lambda_d = (T_{eff}/4 \pi Z_{do} n_{do} e^2)^{1/2} \) and the dust plasma period \( \omega_{pd}^{-1} = (m_d/4 \pi Z_{do} n_{do} e^2)^{1/2} \), respectively. The electrons are assumed to have Boltzmann distribution. Thus, we can express \( n_e \) as

\[ n_e = \frac{1}{\delta_1 + \delta_2 - 1} \exp(\beta_1 s \phi) , \]

where \( \beta_1 = T_{ij}/T_e \), \( s = T_{eff}/T_{ij} \), and \( T_e \) (\( T_{ij} \)) is the thermal energy of electrons (low temperature ions). On the other hand, the ion number densities \( n_{ij} \) in the presence of trapped particles can be expressed as

\[ n_{ij} = \frac{n_{ij}}{n_{do} Z_{do}} \left[ \exp(\Gamma_j) \left( 1 - \text{erf}(\sqrt{\Gamma_j}) \right) + \frac{1}{\sqrt{\beta_j}} \left\{ \exp(\Gamma_j \beta_j) \sqrt{\Gamma_j} \beta_j \right\} \right] , \]

\[ \text{for } \beta_j \geq 0 , \]

\[ \frac{2}{\sqrt{\pi}} \exp(\Gamma_j \beta_j) \int_{0}^{\sqrt{-\Gamma_j} \beta_j} \exp(X^2) dX \text{ for } \beta_j < 0 , \]

where \( \Gamma_j = -T_{eff} \phi/T_{ij} \), \( \beta_j \) represent the ratio of the free ion temperature \( T_{ij} \) to the trapped ion temperature \( T_{ij} \), and \( j = l, h \) for low (high) temperature ions. Now, for \( \phi \ll 1 \), we can approximate \( n_{ij} \) as
charging time is typically of order of $10^2$ milliseconds for micrometer-sized grains,\textsuperscript{10} while the dust characteristic time for dust motion is of order of tens of milliseconds. Thus, using the current balance equation\textsuperscript{4,40} and the hydrodynamic time scale, the dust charge can quickly respond to the plasma having the flat topped and Maxwellian distribution. Therefore, $n_{ij}$ can be rewritten in the form

$$n_{ij} = \frac{n_{ij}}{n_{do} Z_{do}} [\exp(\Gamma_j) - G_j(\Gamma_j)],$$

where $\gamma_j = (1 - \beta_j^2)/\sqrt{\pi}$. The cases $\beta_j = 0$ and $\beta_j = 1$ correspond to the plasma having the flat topped and Maxwellian distributions, respectively. Thus, for isothermal ions we put $b_{kj} = 0$, whereas for the nonisothermal ions, we have $0 < b_{kj} < 1/\sqrt{\pi}$. Therefore, $n_{ij}$ can be rewritten in the form

$$n_{ij} = n_{ij} \frac{\exp(\Gamma_j) - G_j(\Gamma_j)}{n_{do} Z_{do}},$$

where $38,39$ $G_j(\Gamma_j) = \Sigma_{k=1}^{\infty} [2^{(k+1)} b_{kj}(\Gamma_j)^{(2k+1)/2} \Pi(2k+1)].$ Now, substituting $\Gamma_j$ and $j$ into our last equation we can express $n_{ii}$ and $n_{ih}$ as

$$n_{ii} = \frac{\delta_1}{\delta_1 + \delta_2 - 1} \left[ \exp(-s \phi) - G_j(-s \phi) \right],$$

(6)

$$n_{ih} = \frac{\delta_2}{\delta_1 + \delta_2 - 1} \left[ \exp(-s \beta \phi) - G_j(-s \beta \phi) \right],$$

(7)

$\beta_2 = T_{ih}/T_e$ and $\beta = \beta_1/\beta_2$.

We note that $Z_d$ in (2) and (3) is not constant but varies with space and time. Thus, (1)–(3) are completed by the normalized dust grain charging equation.\textsuperscript{40} However, the characteristic time for dust motion is of order of tens of milliseconds for micrometer-sized grains,\textsuperscript{10} while the dust charging time is typically of order of $10^{-6}$ s. Therefore, on the hydrodynamic time scale, the dust charge can quickly reach local equilibrium, at which the currents from the electrons, low- and high-temperature ions to the dust are balanced. Thus, using the current balance equation\textsuperscript{4,40} and the orbit-motion-limited probe model\textsuperscript{41} we have

$$\alpha_1 \delta_1 (1 - s \Psi) \left[ \exp(-s \phi) - G_j(-s \phi) \right] + \alpha_2 \delta_2 (1 - s \beta \Psi) \left[ \exp(-s \beta \phi) - G_j(-s \beta \phi) \right] = \exp(s \beta_1 \Psi + \phi),$$

(8)

where $\Psi = -Z_d Z_{do} e^2/r_{teff}, \alpha_{1,2} = (\beta_{1,2}/\mu_i)^{1/2}$, and $\mu_i = m_i/m_e$. We note that at equilibrium $Z_d = 1$ and $\Psi = \Psi_o = -Z_{do} e^2/r_{teff}$, which can be determined by

$$\alpha_1 \delta_1 (1 - s \Psi_o) + \alpha_2 \delta_2 (1 - s \beta \Psi_o) = \exp(s \beta_1 \Psi_o).$$

(9)

To show the dependence of $Z_d$ on the physical parameters of the system we have numerically analyzed Eq. (8). The numerical results are displayed in Figs. 1 and 2. We have found some significant features in contrast with that studied in earlier works.\textsuperscript{20,22} Figure 1 admits dust charge values for negative potential disturbance that does not appear previously. For positive plasma potential, increasing disturbance strength, first $Z_d$ decreases quickly with a large slope, then gradually slows down with a smaller slope. However, for negative plasma potential disturbance, as its strength increases, $Z_d$ decreases from one (which is the unperturbed dust charge number) to zero. As $\delta_1$ increases, $Z_d$ decreases for the potential disturbance up to the cutoff at $\phi = -3.6$. As $\beta_1$ increases, $Z_d$ increases, while $Z_d$ decreases as $\delta_2$ increases. For a dusty plasma system containing one ion only, $Z_d$ increases rapidly in the neighborhood of the unperturbed electrostatic potential to a greater value than the unperturbed one, thus it behaves like the dusty plasma contains two-temperature ions. Figure 2 shows the effect of the low-temperature trapped ion on the dust charge. We observed that both figures display distinct effects especially for the negative potential values. It is obvious from Fig. 2 that there exist two minimum values for either flat topped distribution or for particles that have velocity less than the thermal velocity.
However, for particles that have velocity greater than the thermal one, $Z_d$ has one minimum and one maximum in the negative potential region.

### III. NONLINEAR DUST ACOUSTIC WAVES

To study the dynamics of small-amplitude DA solitary waves in the presence of adiabatic variation of dust charges and trapped ions of two different temperatures, we employ a reductive perturbation technique. We introduce the stretched coordinates $\xi = e^{\lambda t}(x - \lambda x)$ and $\tau = e^{\lambda t}$, where $\lambda = e$ is a small parameter and $\lambda$ is the solitary wave velocity normalized by $C_d$. The variables $n_d$, $u_d$, $Z_d$ and $\phi$ are expanded as

$$n_d = 1 + \varepsilon n_{d1} + \varepsilon^2 n_{d2} + \varepsilon^3 n_{d3} + \varepsilon^4 n_{d4} + \cdots,$$

$$u_d = \varepsilon u_{d1} + \varepsilon^2 u_{d2} + \varepsilon^3 u_{d3} + \varepsilon^4 u_{d4} + \cdots,$$

$$Z_d = 1 + \varepsilon Z_{d1} + \varepsilon^2 Z_{d2} + \varepsilon^3 Z_{d3} + \varepsilon^4 Z_{d4} + \cdots,$$

$$\phi = \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \varepsilon^3 \phi_3 + \varepsilon^4 \phi_4 + \cdots.$$

Now, substituting these expansions into Eqs. (2)–(8) and collecting the terms of different powers of $\varepsilon$, in the lowest order, we obtain

$$n_{d1} = -R\phi_1,$$

$$u_{d1} = -\lambda R\phi_1,$$

$$Z_{d1} = \gamma_1 \phi_1,$$

where $R = (\lambda^2 - 3\sigma_d)^{-1}$ and

$$\gamma_1 = \frac{\alpha_1 \delta_1(1 + \beta_1)(1 - s \Psi_0) + \alpha_2 \delta_2(\beta + \beta_1)(1 - s \beta \Psi_0)}{\alpha_1 \delta_1(1 + \beta_1(1 - s \Psi_0)) + \alpha_2 \delta_2(\beta + \beta_1(1 - s \beta \Psi_0))}.$$
order to study the nonlinear properties of DA waves. We use the stretched coordinates \( \xi = e^{1/2}(x - \lambda t) \), \( \tau = e^{3/2}t \), and follow the same procedure used before. Accordingly, for the lowest order of \( \varepsilon \) we obtain the relations

\[
n_{d2} = -R \phi_2, \quad u_{d2} = -\lambda R \phi_2, \\
Z_{d2} = \gamma_1 \phi_2 + \gamma_2 (-\phi_1)^{3/2},
\]

(16)

\[
\begin{align*}
\gamma_1 + \frac{s(\beta_1 - \beta_1^2 + \beta_2^2)}{\beta_1 - \beta_1^2 - 1} - R \phi_2 \\
= \left[ \frac{4\beta_2^3 (\beta_1 + \beta_2^2 \delta_1 + \beta_2^2 \delta_2^2)}{3(\beta_1 + \beta_2 - 1)} - \gamma_2 \right] (-\phi_1)^{3/2}.
\end{align*}
\]

(17)

For the next order in \( \varepsilon, O(\varepsilon^2) \), we obtain a set of equations, which, after making use of Eqs. (11) and (16), yields

\[
\frac{\partial \phi_1}{\partial \tau} + A \frac{\partial}{\partial \xi} \left( (-\phi_1)^{1/2} \phi_2 \right) + B \frac{\partial^3 \phi_1}{\partial \xi^3} + C \frac{\partial \phi_1}{\partial \xi} = 0,
\]

(18)

where

\[
C = B \left[ \frac{s^2 (\beta_1 - \beta_1^2 + \beta_2^2 \delta_2)}{\beta_1 - \beta_1^2 - 1} + 3R (\gamma_1 - [\lambda^2 + \sigma_d]R^2) \
- 2\gamma_3 \right]
\]

and

**FIG. 3.** \( \phi_{1m} \) is plotted against \( \delta_2 \) for different values of \( \delta_1 \), where \( \beta_1 = 0.0001, \beta_2 = 0.1, \sigma_d = 0, \beta_i = 0.2, \) and \( \beta_b = 0.8 \).

**FIG. 4.** \( \phi_{1m} \) is plotted against \( \beta_1 \), where \( \delta_1 = 1, \delta_2 = 5, \beta_1 = 0.0001, \beta_2 = 0.1, \sigma_d = 0, \) and \( \beta_b = 0.8 \).

**FIG. 5.** \( \phi_{1m} \) is plotted against \( \beta_1 \), where \( \delta_1 = 1, \delta_2 = 5, \beta_1 = 0.0001, \beta_2 = 0.1, \sigma_d = 0, \) and \( \beta_b = 0.8 \).

**FIG. 6.** \( \lambda \) and width are plotted against \( \sigma_d \), where \( \delta_1 = 1, \delta_2 = 10, \beta_1 = 0.1, \beta_2 = 0.5, \beta_i = 0.2, \) and \( \beta_b = 0.8 \).
only differences between Eq. \( I \)

\[ \gamma_1 = \frac{\gamma_d}{\gamma_o}, \quad \gamma_d = \gamma_{d1} + \gamma_{d2} + \gamma_{d3}, \]

\[ \gamma_{d1} = \frac{s}{2} \left\{ \alpha_1 \delta_1 (1 - \beta_1^2) (1 - s \Psi_o) + \alpha_2 \delta_2 (\beta^2 - \beta_1^2) \right\} \times (1 - s \beta \Psi_o), \]

\[ \gamma_{d2} = \gamma_1 s \Psi_o \left[ \alpha_1 \delta_1 (1 - \beta^2_1) (1 - s \Psi_o) \right] + \alpha_2 \delta_2 (\beta^2 - \beta_1^2) (1 - s \beta \Psi_o), \]

\[ \gamma_{d3} = -s (\gamma_1 \beta_1 \Psi_o)^2 \left[ \alpha_1 \delta_1 (1 - s \Psi_o) \right] + \alpha_2 \delta_2 (1 - s \beta \Psi_o). \]

The one-soliton solution of Eq. (18), \( A = 0 \), is given by

\[ \phi_1 = \phi_{2m} \text{sech}^2 \left[ \eta/w_2 \right], \]  

where the amplitude \( \phi_{2m} \) and the width \( w_2 \) are given by \( 3M/C \) and \( 2 \sqrt{B/M} = 0.5 \ w_1 \), respectively. Since \( \gamma_1 \geq 0, \gamma_2 \geq 0 \) and \( M > 0 \), Eq. (19) clearly indicates that both rarefactive and compressive solitons exist. The width of the soliton becomes narrower than that of the MKdV equation. One can observe that the inclusion of the second trapped ion species admits the existence of the two kinds of solitons. On the other hand, when \( A \phi_2 \to -2D \phi_1/3 \), Eq. (18) would reduce to

\[ \frac{\partial \phi_1}{\partial \tau} + D (- \phi_1)^{1/2} \frac{\partial \phi_1}{\partial \xi} + B \frac{\partial^3 \phi_1}{\partial \xi^3} + C \phi_1 \frac{\partial \phi_1}{\partial \xi} = 0. \]  

Equation (20) has the same form as Eq. (49) of Ref. 34, the only differences between Eq. (20) here and Eq. (49) of Ref. 34 are in coefficients \( D, B, \) and \( C \). If \( B \) is \( O(1) \), Eq. (20) reduces to Eq. (49) of Ref. 34, and the \( \frac{1}{2} \) scaling prevails accounting for a balance of the increased nonlinearity and dispersion. On the other hand, if \( B \) is \( O(e^{1/2}) \), the nonlinearity is weakened and becomes comparable to the ordinary hydrodynamic nonlinearity represented by the C-term, and the \( \frac{1}{4} \) scaling is appropriate. The scaling, hence, reflects the strength of the nonlinearity accounting for different solitary wave solutions. The most general solitary wave solution in which both nonlinearities are taken into account is given by Eq. (48) of Ref. 34. Now, substituting \( \eta = \xi - M \tau \) in Eq. (20) and integrating twice, using (14), we get

\[ \frac{1}{2} \left( \frac{d \phi_1}{d \eta} \right)^2 = \frac{M}{2B} \left( \frac{\Phi^2}{2B} \right) \left( 1 - \frac{8D (- \phi_1)^{1/2}}{15M} - \frac{C \phi_1}{3M} \right) \]

\[ = -V(\phi_1, M). \]  

Hence

\[ V(\phi_1, M) = -\frac{M \phi_1^2}{2B} + \frac{4D (- \phi_1)^{5/2}}{15B} + \frac{C \phi_1^3}{6B}. \]  

For the formation of DLs, we must satisfy the following conditions:

\[ V(\phi_m, M) = 0, \quad \left( \frac{dV}{d \phi_1} \right)_{\phi_1 = \phi_m} = 0 \text{ and} \]

\[ \left( \frac{d^2V}{d \phi_1^2} \right)_{\phi_1 = \phi_m} < 0. \]  

Using \( y^2 = -\phi_1 \) in Eq. (22), we have

\[ V(y, M) = -\frac{M y^4}{2B} + \frac{4D y^5}{15B} - \frac{C y^6}{6B}, \]

which can be rewritten as

\[ V(y) = -\frac{C y^4}{6B} (y - y_m)^2. \]

Substituting \( M \) and \( D \) into the relation (21), we can transform into

\[ \frac{d^2y}{d \eta^2} = \frac{C y^2}{12B} (y - y_m)^2. \]

Thus, the DL solution is

\[ \phi_1 = -\frac{y_m^2}{4} \left[ 1 - \tanh(\eta/w) \right]^2, \]

where

\[ w = \frac{5}{D \sqrt{3BC}}. \]

Here, rarefactive DA DLs are admitted only contrary to the case studied in recent studies with the inclusion of ion beam or trapped electrons.

Now, if the nonlinear coefficient of the KdV equation vanishes, \( C = 0 \), i.e., the KdV equation fails to describe the system successfully. This force us to look for another equation which is suitable for describing the evolution of the system. Figure 7 shows the relation between \( \beta_1 \) and \( \beta_2 \) corresponding to two values of \( \beta_1 \) and \( C = 0 \). It shows that \( \beta_2 \) increases as \( \beta_1 \) or \( \beta_1 \) increases. Instead of the stretching used before we have to use higher stretching coordinates of the perturbation theory, \( \xi = e^{3M} (x - \lambda t) \) and \( \tau = e^{9M} t \). We obtain the linear relation, Eq. (12), for the lowest order, and for the next orders of \( e \) we get the same relations (11), (16) and (17). The third order perturbed quantities can be obtained as

\[ n_{d3} = R \{- \phi_3 + \Phi_1 \phi_3^2, u_{d3} = \lambda R \{- \phi_3 + \Phi_1 \phi_3^2, \}

\[ Z_{d3} = \gamma_1 \phi_3 + \frac{i}{2} \gamma_2 \phi_2 (- \phi_1)^{1/2} + \gamma_3 \phi_1^2, \]  

and
where

\[ \phi_3 = \left[ 1 + \frac{s(\beta_1 + \delta_1 + \delta_4)}{2(\delta_1 + \delta_2 - 1)} R \right]^{-1} \]

\[ \times \left[ \frac{s^2(\delta_1 - \beta^2 + \beta^2 \delta_2)}{2(\delta_1 + \delta_2 - 1)} + \frac{3 R(\gamma_1 - [\lambda^2 + \sigma_4]R^2)}{2} \right] \phi_1^2 \]

\[ + \frac{2 s \delta_2 (\delta_1 b_{11} + \delta_2 b_{12} \beta^{3/2})}{(\delta_1 + \delta_2 - 1)} - 3 \phi_2 \phi_1 \phi_1^{1/2} \phi_2 \right], \]

where

\[ E_1 = -\frac{1}{2} \gamma_1 + \frac{1}{2}[\lambda^2 + \sigma_4]R^2, \quad E_2 = E_1 - \lambda R^2. \]

If we continue to the next order of \( \varepsilon \), we get a system of equations in the subscripted 4 perturbed quantities. Eliminating the perturbed quantities with subscript 4, we obtain a further MKdV (FMKdV) equation

\[ \frac{\partial \phi_1}{\partial \tau} + F(-\phi_1)^{3/2} \frac{\partial \phi_1}{\partial \xi} + B \frac{\partial^3 \phi_1}{\partial \xi^3} = 0, \tag{29} \]

where

\[ F/B = \frac{1}{5} \left[ 21 \gamma_2 R - 15 \gamma_4 + \frac{8 s \delta_2 (\delta_2 b_{21} + \delta_2 b_{22} \beta^{3/2})}{(\delta_1 + \delta_2 - 1)} \right]. \]

\[ \gamma_4 = \frac{\gamma_e}{\Psi_e}, \quad \gamma_e = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4, \]

\[ \gamma_1 = -\frac{s}{2 \pi \sqrt{\delta_1}} \left[ \alpha_1 \delta_1 (1 - s \Psi_e) b_{21} + \alpha_2 \delta_2 b_{22} (1 - s \beta \Psi_o) \right], \]

\[ \gamma_2 = -\frac{s}{2 \pi \sqrt{\delta_1}} \left[ \alpha_1 \delta_1 b_{11} + \alpha_2 \delta_2 b_{21} \right], \]

The one-soliton solution of Eq. (29) is given by

\[ \phi_1 = \phi_{1m} \text{sech}^{1/2}(\eta/w_3), \tag{30} \]

where the amplitude \( \phi_{1m} \) and the width \( w_3 \) are given by \((35M/8F)^{1/3}\) and \((\tilde{\eta}B/M)^{1/3}\), respectively.

Figures 8 and 9 show the variation of \( \phi_{1m} \) for different parametric regimes. It is shown that (i) \( \phi_{1m} \) decreases rapidly as \( \beta_1 \) increases for \( \beta_1 < 10^{-3} \), then it decreases gradu-
ally with small slope for $\beta_1 > 10^{-3}$. (ii) It decreases rapidly as $\beta_1$ increases. Also, $\phi_{3m}$ has the same features as $\phi_{1m}$ corresponding to the same variation of $\delta_1$, $\delta_2$ and $\sigma_d$ but with a larger rarefactive amplitude (not shown).

IV. DISCUSSION

We have studied the effects of the dust charge variation on the small-amplitude DA waves in dusty plasmas having two-temperature trapped ions. The modified KdV equation and FMKdV equation with high-order nonlinear terms, as well as the DL solution, are also obtained. We have found that the large dust-charge fluctuation induced by the low-temperature ions is the physical reason why two-temperature-ions dusty plasmas can admit the transition of soliton and/or solitary waves for dust density profile from the rarefactive type to the compressive type when the system parameter changes. It is also found that the presence of lower-temperature ions plays a crucial role in the coexistence of both compressive and rarefactive waves, as well as the DL. Our numerical results also confirm that the positive plasma potential soliton should exist in the dusty plasma with relative low-temperature or/and small number density regime of lower-temperature ion.

It is found that the soliton width becomes narrower compared with the case of constant dust charge. The modification introduced by the variable dust charge on the amplitude and width of the solitons are analyzed. Finally, it is pointed out that the approximate similarity law that El-Labany and El-Taibany predicted$^{20}$ is no longer valid for the present system.

The results that we obtained from this investigation may be summarized as follows: (i) As $\delta_1$ increases, $Z_d$, $\phi_{1m}$, rarefactive $\phi_{2m}$ and $\phi_{3m}$ decrease; but $\lambda$ and compressive $\phi_{2m}$ increase. (ii) As $\delta_2$ increases, $\lambda$, $Z_d$ and compressive $\phi_{2m}$ decrease; however, $\phi_{1m}$, rarefactive $\phi_{2m}$ and $\phi_{3m}$ increase. (iii) As $\beta_1$ increases, $Z_d$ and compressive $\phi_{2m}$ increase but rarefactive $\phi_{3m}$ decreases. $\phi_{1m}$ first increases very rapidly for $\beta_1 < 10^{-3}$; then, for $\beta_1 > 10^{-3}$, it decreases with a smaller slope. On contrary, $\phi_{3m}$ decreases very rapidly for $\beta_1 < 10^{-3}$, then the decrement slope becomes smaller one. (iv) As $\beta_1$ increases, $\phi_{1m}$ and $\phi_{3m}$ decrease. $Z_d$ is strongly affected, especially for negative potential region and through the transition from subthermal particles to superthermal ones. (v) As $\sigma_d$ increases, $\lambda$, $\phi_{1m}$, compressive $\phi_{2m}$ and $\phi_{3m}$ increase but rarefactive $\phi_{3m}$, and the soliton width decrease.

This work agrees exactly with the results of Xie et al.$^{22}$ by neglecting the dust temperature and trapped ions and also with the results of Mamun$^{31}$ on neglecting the nonisothermal parameters. It may be added that the effects of obliqueness, external magnetic field and inhomogeneity in plasma density on these solitary structures, and their instabilities, are also important problems, but beyond the scope of our present work.

ACKNOWLEDGMENTS

The authors are grateful to Professor H. Schamel for his various critical suggestions and continuous investigations. Also, our appreciation goes to the referee for a number of valuable criticisms and comments that have led to improvement of the original manuscript. The authors also thank the editor and his staff for their kind assistance.

4P. K. Shukla and A. A. Mamun, Introduction to Dusty Plasma Physics (Institute of Physics, Bristol, 2002).
22B. Xie, K. He, and Z. Huang, Phys. Plasmas 6, 3808 (1999).