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On the quaternion projective space

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ABSTRACT
Apart from being a vital and exciting field in mathematics with interesting results, projective spaces have various applications in design theory, coding theory, physics, combinatorics, number theory and extremal combinatorial problems. In this paper, we consider real, complex and quaternion projective spaces. We focus on the geometric feature of the sectional curvatures. We first study the real and complex projective spaces. We prove that their sectional curvatures are constant. Then, we consider the quaternion projective space. Specifically, we prove that the quaternion projective space has a positive constant sectional curvature. *We also determine the pinching constant for the complex and quaternion projective spaces.*

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1. Introduction

The projective geometry plays a vital role in many visual computing domains, in particular, computer vision modelling and computer graphics [1]. It gives a mathematical formalism to prescribe the geometry of cameras and the associated transformations, therefore enabling the design of computational approaches that manipulate 2-dimensional (2-D) projections of 3-D objects. In this regard, a fundamental side is the fact that objects at infinity can be represented and handled with projective geometry and this in contrast to the Euclidean geometry. Indeed the projective geometry turns out to be very useful in order to prescribe some complex phenomena in physics [2].

The quaternions $\mathbb{Q}$ be 4-D real algebra generated by the identity element 1 and the symbols i, j and k. So

$$\mathbb{Q} = \{q_0 + q_1 i + q_2 j + q_3 k : q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

where

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \\ i^2 = j^2 = k^2 = -1. \tag{1}$$

We call $\operatorname{Re} q = q_0$ the real part of the quaternion number q and $\operatorname{Im} q = q - q_0$ the imaginary part. The space of imaginary $q \in \mathbb{Q}$ is denoted $\operatorname{Im}\mathbb{Q}$. The conjugate of q is $\bar{q} = \operatorname{Re} q - \operatorname{Im} q = q_0 - q_1 i - q_2 j - q_3 k$. The norm of q, $|q| = \sqrt{q\bar{q}} = \sqrt{\bar{q}q} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$. Recall that $\mathbb{Q}$ is not commutative, so that we should be careful with the order of factors in products. Indeed $\mathbb{Q}$ is a real division algebra i.e. $|pq| = |p||q|$ for all $p, q \in \mathbb{Q}$. $\mathbb{Q}^n$ is the $n-D$ right module over the quaternions $\mathbb{Q}$. If $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are two vectors in $\mathbb{Q}^n$, where $a_i, b_i \in \mathbb{Q}$, then we denote by aq the vector $(a_1 q, a_2 q, \ldots, a_n q)$, $q \in \mathbb{Q}$.

The set of all matrices of degree n with coefficients in $\mathbb{Q}$ will be denoted by $M(n, \mathbb{Q})$. A matrix $\sigma \in M(n, \mathbb{Q})$ is said to be symplectic if and only if $\bar{\sigma}^t \cdot \sigma = I$, where I is the identity $n \times n$ matrix, that is $\sigma^{-1} = \bar{\sigma}^t$, σ^{-1} being the inverse of σ defined by $\sigma^{-1} \cdot \sigma = I$. The group of all symplectic matrices will be denoted by $Sp(n)$, we denote the Lie algebra of $Sp(n)$, the set of matrices $\sigma \in M(n, \mathbb{Q})$ which satisfies $\bar{\sigma} + \sigma^t = 0$, by $sp(n)$.

Quaternion Algebra is a bit of a mixed bag, this is a very interesting and powerful tool for both modelling certain phenomenon, and for algebraic study. Moreover it turns out to be very important in areas of mesh deformation, biomechanics, physics, computer graphics and molecular modelling. For more details about the quaternion algebra, we refer to [3].

Riemannian submersions, presented by O'Neill [4] and Gray [5], have been utilized by various authors to establish some definite Riemannian metrics, such as Einstein, positively curved [6,7]. Indeed Riemannian submersions are used to investigate different geometric structures of Riemannian manifolds [8,9]. Sectional curvature is of great importance in differential geometry. This concept describes how curved the space is in some 2-D subspace of the tangent space at a given point.

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We are interested in those homogeneous spaces that have strictly positive sectional curvatures. An important feature of these spaces is the so called "pinching constant" i.e. the quotient of the maximal and minimal positive sectional curvatures.

This article is concerned with a study of some geometric properties of the projective spaces as a Riemannian homogenous symmetric spaces. The geometric feature that we focus on are the sectional curvatures. Namely, we determine the sectional curvature for the real projective spaces, using the properties of it as a homogenous Riemannian symmetric space. We also determine the sectional curvature for the complex and quaternion projective spaces, using the Riemannian submersions and O'Neill formula. An important feature of these spaces is called pinching constant, which is the quotient of the maximal and minimal positive sectional curvatures. The projective spaces can be studied as a separate field, but are also used in different applied areas, geometry especially. The projective spaces play an important role in various aspects combinatorics, design theory, number theory, physics, coding theory and extremal combinatorial problems. Indeed projective spaces are important for topology and algebraic topology as well. There are differential projective spaces that finite projective spaces that have applications in analysis and discrete mathematics. Many authors mainly had paid attention to study the projective spaces and their applications, see [10–15].

2. Preliminaries

We represent the set of all tangent vectors at p by $T_p(M)$, the so called tangent space, where M is smooth manifold, $\chi(M)$ is the set of all smooth vector fields and $C^\infty(M)$ is the set of smooth functions of M.

Definition 2.1 ([16]): A connection ∇ on a smooth manifold M is a map $\nabla : \chi(M) \times \chi(M) \mapsto \chi(M), \quad (E, F) \mapsto \nabla_E F$, which satisfies the following properties:

(a) $\nabla_E(F + \Upsilon) = \nabla_E F + \nabla_E \Upsilon$,
(b) $\nabla_{E+F}\Upsilon = \nabla_E \Upsilon + \nabla_F \Upsilon$,
(c) $\nabla_{hE} F = h\nabla_E F$,
(d) $\nabla_E hF = E(h)F + h\nabla_E F, \forall \; E, F, \Upsilon \in \chi(M)$ and $h \in C^\infty(M)$.

Definition 2.2 ([17]): A Riemannian metric g on a smooth manifold M is a tensor of type $(0, 2)$ that obey

(a) $g(E, F) = g(F, E)$,
(b) $g(\alpha E + \gamma F, \Upsilon) = \alpha\, g(E, F) + \gamma\, g(F, \Upsilon)$,
(c) If $g(E, F) = 0, \forall E \in \chi(M)$, then $F = 0$,
(d) $g(E, E) > 0, \forall \; E \neq 0, E, F, \Upsilon \in \chi(M)$ and $\alpha, \gamma \in C^\infty(M)$.

(M, g) is called a Riemannian manifold.

Definition 2.3 ([17]): A pseudo-Riemannian manifold is a pair (M, g), M is a real differentiable manifold and g is a field of non-degenerate symmetric bilinear forms on M.

Proposition 2.1 ([16]): *Consider a Riemannian manifold M, there is a unique connection ∇ on M satisfies*

(1) $\nabla_E F - \nabla_F E = [E, F]$.
(2) $E \cdot g(F, \Upsilon) = g(\nabla_E F, \Upsilon) + g(F, \nabla_E \Upsilon), \qquad E, F, \Upsilon \in \chi(M)$.

Here ∇ is called Riemannian connection or Levi–Civita connection.

Definition 2.4 ([8]): Consider (M, g) and ∇ are the Riemannian manifold and Riemannian connection, respectively. The curvature tensor of type $(1, 3)$ defined by

$$R(E, F)\Upsilon = \nabla_E \nabla_F \Upsilon - \nabla_F \nabla_E - \nabla_{[E,F]}\Upsilon.$$

Definition 2.5 ([17]): Given a point $p \in M$ and let $l \in T_p(M)$ be a 2-D subspace of $T_p(M)$ and let $E, F \in l$ be two linearly independent vectors. Then

$$\mathcal{K}(E, F) = \frac{g(R(E, F)F, E)}{g(E, E)g(F, F) - g(E, F)^2} \tag{2}$$

does not depend on the choice of the vectors $E, F \in l$ and $\mathcal{K}(E, F)$ is called the sectional curvature of l at p, where R is the curvature tensor at p. If all sectional curvature at all points of M is equal to constant C, then M is said to be a space of constant curvature.

Remark 2.1: If E and F are orthonormal vectors, then the sectional curvature of a Riemannian manifold is denoted by

$$\mathcal{K}(E, F) = g(R(E, F)F, E).$$

Definition 2.6 ([18]): Let V be a finite-dimensional vector space over an arbitrary field K. The projective space $\mathbb{P}(V)$ is the set of $1-D$ linear subspaces of V, where $\dim \mathbb{P}(V) = \dim V - 1$ (dimension of $\mathbb{P}(V)$), which denoted by $\mathbb{P}^n_K$ or $\mathbb{KP}^n$.

Definition 2.7: The standard unit n-sphere S^n is the set of points $(x_0, x_1, \ldots, x_n)$ in $\mathbb{R}^{n+1}$, which obey the equation $x_0^2 + x_1^2 + \cdots + x_n^2 = 1$.

3. Riemannian submersions

In this section, we need some important input from [4,5] and [6, Chapter 9] about the Riemanniana submersions and the most important related notions for our purpose.

Definition 3.1 ([4]): Let $\pi : M \to B$ be a differentiable map between differential manifolds M and B. Then π

is a submersion if its differential $d\pi_p$ is surjective for all points $p \in M$.

Definition 3.2 ([4]): Let $\pi : (M, g) \to (B, f)$ be a smooth submersion. Then the vertical bundle $\mathcal{V} = \mathrm{Ker}(d\pi)$ is the kernel of the differential $d\pi$. The horizontal bundle is the orthogonal complement to $\mathcal{V}$, i.e. $\mathcal{H} = (\mathcal{V})^{\perp}$.

At each point $p \in M$, we have $T_pM = \mathcal{V}_p \oplus \mathcal{H}_p$. A vector field on a manifold M is said to be vertical (or horizontal) if it is tangent (or orthogonal) to fibres $\pi^{-1}(b)$, for all $b \in B$. We also denote the projections of vector fields in $C^{\infty}(TM)$ to the vertical and horizontal bundles by $\mathcal{V}$ and $\mathcal{H}$, respectively. The horizontal and vertical parts of vector field X on M are represented by $X^{\mathcal{H}}$ and $X^{\mathcal{V}}$, respectively.

Definition 3.3 ([4]): Consider (M, g) and (B, f) are Riemannian manifolds, p be a point in M. A Riemannian submersion $\pi : M \to B$ is a mapping with a differential $d\pi$ that satisfies

(1) $d\pi : T_pM \to T_{\pi(p)}B$ is surjective for all $p \in M$,
(2) $d\pi$ preserves lengths of horizontal vectors, i.e.

$$g(X, Y) = f(d\pi(X), d\pi(Y)),$$

$$\text{for all horizontal } X \text{ and } Y.$$

O'Neill in [4] defines a fundamental tensor describes submersion as follows: For arbitrary vector fields $E, F \in C^{\infty}(TM)$ on M, the tensors A is defined as

$$A_EF = \mathcal{V}\nabla_{\mathcal{H}E}(\mathcal{H}F) + \mathcal{H}\nabla_{\mathcal{H}E}(\mathcal{V}F). \tag{3}$$

Lemma 3.1 ([4]): *If X and Y are horizontal vector fields, then $A_XY = -A_YX = \frac{1}{2}\mathcal{V}[X, Y]$.*

Definition 3.4 ([19]): Let (M, g) and (B, f) be two pseudo-Riemannian manifolds. A smooth surjective submersion $\pi : M, \to B$ is a pseudo-Riemannian submersion (see [20]) when $d\pi$ preserves scalar products of vectors normal to fibres and when the metric induced on every fibre $\pi^{-1}(b)$, where $b \in B$, is non-degenerate.

Proposition 3.1 ([4]): *If X, Y, Z, H be horizontal vector fields on M. Then the curvatures R of M and R^* of B satisfy*

$$\langle R^*(X, Y)Z, H\rangle = \langle R(X, Y)Z, H\rangle + 2\langle A_XY, A_ZH\rangle - \langle A_YZ, A_XH\rangle - \langle A_ZX, A_YH\rangle. \tag{4}$$

Corollary 3.1 ([4]): *Let $\pi : M \to B$ be a submersion, and let $\mathcal{K}$ and $\mathcal{K}^*$ be the sectional curvature of M and B, respectively. If X and Y are horizontal vectors at a point of M, then*

$$\mathcal{K}^*(X, Y) = \mathcal{K}(X, Y) + 3|A_XY|^2. \tag{5}$$

4. Real and complex projective spaces

We discuss some important as well as interesting properties for real and complex projective spaces. First, we give some definitions and propositions, which turn out to be very important in order to study these projective spaces in a completely unified way. Second, we prove that these projective spaces are spaces of constant curvature.

It is well known that the Grassmann manifolds $G_{p,q}(\Psi)$ of all p-planes in Ψ^{p+q}, where Ψ is the set of real numbers, complex numbers or quaternions. As a special case $G_{1,q}(\Psi)$ or $G_{p,1}(\Psi)$ is a projective space. For more details about Grassmannians manifold, we refer to [16,21].

4.1. Real projective space $\mathbb{RP}^n$

Here we determine the sectional curvature for the real projective spaces, using the properties of it as a homogenous Riemannian symmetric space.

Definition 4.1: $\mathbb{RP}^n$ is the set of all 1-D subspaces through the origin in $\mathbb{R}^{n+1}$.

We define an equivalence relation $\sim$ on $\mathbb{R}^{n+1}\backslash\{0\}$ as, $a \sim b \Leftrightarrow a = \mu b$ for some $\mu \in \mathbb{R} - \{0\}$. The quotient space (set of all equivalence classes) is precisely $\mathbb{RP}^n$.

Since each line through the origin in $\mathbb{R}^{n+1}$ intersects the sphere S^n, we can keep under control this relation to S^n:

$$a, b \in S^n; a \sim b \Leftrightarrow a = \mu b$$

$$\text{for some } \mu \in \mathbb{R} \text{ with } |\mu| = 1.$$

Let $\pi : S^n \to \mathbb{RP}^n$ be the quotient map, which assigns to $a \in S^n$ the line in $\mathbb{R}^n$ through a, and let $\pi(a) = [a]$. The inverse image $\pi^{-1}([a])$ of any point $[a] \in \mathbb{RP}^n$ is the two point set $\{a, -a\}$, which is isomorphic to the 0-sphere S^0.

Proposition 4.1 ([22]): $\mathbb{RP}^n \cong S^n/\{\pm 1\}$.

Remark 4.1: Every point in $\mathbb{RP}^n$ is depicted by two points in S^n.

Definition 4.2: Let G be a Lie group and K be a closed subgroup with Lie algebras $\underline{g}$ and $\underline{h}$. A homogeneous space G/K is called reductive if there exists a complementary subspace $\underline{m}$ of $\underline{h}$ in $\underline{g}$ that is $Ad(K)$-invariant i.e. $\underline{g} = \underline{h} \oplus \underline{m}$ with $Ad(H)(\underline{m}) \subset \underline{m}$.

Proposition 4.2: *The real projective space has constant sectional curvature with value 1.*

Proof: Consider the real projective space denoted by $G_{1,q}(\mathbb{R})$ or $G_{p,1}(\mathbb{R})$. let

$$X, N \in \underline{m} = \left\{ W = \begin{pmatrix} 0 & \mathcal{A} \\ -\mathcal{A}^t & 0 \end{pmatrix}; \mathcal{A} \text{ is } p \times q \text{ real matrix} \right\}$$

be the orthonormal vectors. Hence the sectional curvature of the plane spanned by X, N, is specified by Equation (2). If $q = 1$,

$$N = \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_N \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_N \end{pmatrix}$$

are orthonormal vectors i.e.

$$g(N,N) = g(X,X) = 1 \quad \text{and} \quad g(X,N) = 0,$$
$$\text{then } \mathcal{K}(X,N) = -g(R(N,X)N, X).$$

Now we want to prove that $\mathcal{K}(X,N)$ is constant. Consider the inner product

$$g(A,B) = \operatorname{Re}\operatorname{tr} A\,B^t. \tag{6}$$

The curvature tensor R is given by

$$R(X,Y)Z = XY^tZ + ZY^tX - YX^tZ - ZX^tY, \tag{7}$$

where X, Y, Z are real $p \times q$ matrices.

From Equations (6) and (7) we get

$$\begin{aligned} R(N,X)N &= 2T(N,X,N) - T(X,N,N) - T(N,N,X) \\ &= 2NX^tN - XN^tN - NN^tX, \end{aligned}$$

where $T(X,Y,Z) = XY^tZ$ and as $g(X,N) = 0, g(N,N) = 1$, we obtain

$$NX^tN = \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_p \end{pmatrix} \begin{pmatrix} x_1 & . & . & . & x_p \end{pmatrix} \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_p \end{pmatrix}$$

$$= \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_p \end{pmatrix} (x_1a_1 + \cdots + x_pa_p) = 0,$$

$$XN^tN = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_p \end{pmatrix} \begin{pmatrix} a_1 & . & . & . & a_p \end{pmatrix} \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_p \end{pmatrix} = \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_p \end{pmatrix} = X,$$

$$NN^tX = \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_p \end{pmatrix} \begin{pmatrix} a_1 & . & . & . & a_p \end{pmatrix} \begin{pmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_p \end{pmatrix} = 0,$$

then $R(N,X)N = 0 - X - 0 = -X$. It follows that

$$\mathcal{K}(X,N) = -g(R(N,X)N, X) = \langle X, X \rangle = 1.$$

Hence $G_{p,1}(\mathbb{R})$ is a space of constant curvature, similarly for $G_{1,q}(\mathbb{R})$. ■

4.2. Complex projective space $\mathbb{CP}^n$

In this section, we determine the sectional curvature for the complex projective space, using the Riemannian submersions and O'Neill formula.

Definition 4.3: $\mathbb{CP}^n$ is the set of 1-D complex–linear subspaces of $\mathbb{C}^{n+1}$.

We define an equivalence relation $\sim$ on $\mathbb{C}^{n+1}\backslash\{0\}$ as $a \sim b \Leftrightarrow a = \mu b$. The quotient space is exactly $\mathbb{CP}^n$.

Since each line through the origin in $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$ intersects the sphere S^{2n+1}. We can keep under control this relation to S^{2n+1}:

$$a, b \in S^{2n+1}; a \sim b \Leftrightarrow a = \mu b$$
$$\text{for some } \mu \in \mathbb{C} \text{ with } |\mu| = 1.$$

Let $\pi : S^{2n+1} \to \mathbb{CP}^n$ be the quotient map, which assigns to $a \in S^{2n+1}$ the complex line in $\mathbb{C}^{n+1}$ through a, and let $\pi(a) = [a]$. The inverse image $\pi^{-1}([a])$ of any point $[a] \in \mathbb{CP}^n$ is the set $\{e^{i\theta}a; \theta \in [0, 2\pi]\}$, which is isomorphic to the 1-sphere S^1.

Proposition 4.3 ([22]): $\mathbb{CP}^n \cong S^{2n+1}/S^1$.

Remark 4.2: Each point in $\mathbb{CP}^n$ is represented by a circle in S^{2n+1}.

Proposition 4.4: *The complex projective space has sectional curvature lies in interval* $[1, 4]$.

Proof: Let us consider the Hopf bundle $\pi : S^{2n+1} \to \mathbb{CP}^n$ and N be the unit normal on the unit sphere $S^{2n+1} \subseteq \mathbb{R}^{2n+2}$, then IN is defined the vertical vector field on S^{2n+1}. Then

$$A_XY = \langle X, IY \rangle IN, \quad A_X(IN) = IX,$$

where X and Y are horizontal vectors on S^{2n+1} and I is complex structure. Since IN is a unit field spanning the vertical distribution, hence

$$\begin{aligned} |A_XY|^2 &= \langle A_XY, IN \rangle^2 = \langle Y, A_X(IN) \rangle^2 \\ &= \langle Y, IX \rangle^2 = \langle X, IY \rangle^2. \end{aligned} \tag{8}$$

For the orthonormal vector fields X and Y, it follows from Equations (5) and (8) that

$$\mathcal{K}^*(X,Y) = 1 + 3|A_XY|^2 \qquad = 1 + 3\langle X, IY \rangle^2.$$

Thus, the sectional curvature of the complex projective space $\mathbb{CP}^n$ lies between 1 and 4. ■

Remark 4.3: The projective spaces $\mathbb{RP}^n$ and $\mathbb{CP}^n$ are compact, Hausdorff, second countable and smooth manifolds of dimensions n & $2n$.

In the next section, we study widely from one of the most important type of projective spaces, namely the quaternion projective space.

5. Quaternion projective space

We prove that the quaternion projective space is a space of constant curvature, using the Riemannian submersions and O'Neill formula.

Definition 5.1: The quaternion projective space $\mathbb{QP}^n$ is the set of all 1-D subspaces through the origin in $\mathbb{Q}^{n+1}$. It is a compact, smooth 4n-D manifold

Consider the quaternion projective space $\mathbb{QP}^n$ as the set of all (unordered) directions in $\mathbb{Q}^{n+1}$. A quaternion line is isomorphic to $\mathbb{R}^4$, but not all real 4-D subspaces of $\mathbb{Q}^{n+1}$ are complex lines. We define an equivalence relation $\sim$ on $\mathbb{Q}^{n+1}\backslash\{0\}$ by $a \sim b \Leftrightarrow a = \mu b$ for some $\mu \neq 0$ in $\mathbb{Q}$. The quotient space is exactly $\mathbb{QP}^n$.

Since each line through 0 in $\mathbb{Q}^{n+1} \cong \mathbb{R}^{4n+4}$ intersects the sphere S^{4n+3}, we can keep under control this relation to S^{4n+3}:

$$a, b \in S^{4n+3}; a \sim b \Leftrightarrow a = \mu b \quad \text{for some } \mu \in \mathbb{Q} \quad \text{with } |\mu| = 1.$$

Let $\pi : S^{4n+3} \to \mathbb{QP}^n$ be the quotient map (Hopf map), which assigns to $a \in S^{4n+3}$ the quaternionic line in $\mathbb{Q}^{n+1}$ through x, and let $\pi(a) = [a]$. The inverse image $\pi^{-1}([a])$ of any point $[a] \in \mathbb{QP}^n$ is the set $\{qa; q \in \mathbb{Q}, |q| \neq 1\}$, which is isomorphic to the 1-sphere S^3.

Remark 5.1: Each point in $\mathbb{QP}^n$ is represented by a 3-sphere in S^{4n+3}.

Definition 5.2 ([23]): Let M be the quaternion projective space, $x \in M$. For each two unit vectors X, Y in $T_x(M)$, define the "angle" function $\varphi(X, Y)$, $0 < \varphi(X, Y) < \frac{\pi}{2}$ as follows

$$\cos^2 \varphi(X, Y) = \langle XI, Y\rangle^2 + \langle XJ, Y\rangle^2 + \langle XK, Y\rangle^2. \tag{9}$$

φ is well defined because it is independent of the choice of a quaternionic structure $I, J, K : I^2 = J^2 = K^2 = -1$ and $IJ = K$ on $T_x(M)$.

Proposition 5.1: *The quaternion projective space has sectional curvature lies in interval* $[1, 4]$.

Proof: Let us consider the Hopf bundle $\pi : S^{4n+3} \to \mathbb{QP}^n$ and N be the unit normal on the unit sphere $S^{4n+3} \subseteq \mathbb{R}^{2n+2}$, then IN, JN and KN are defined the vertical vector fields on S^{4n+3}. Hence

$$A_X Y = -\langle IX, Y\rangle IN - \langle JX, Y\rangle JN - \langle KX, Y\rangle KN, \tag{10}$$

where X, Y are horizontal vector of the Hopf bundle and I, J, K are complex structure with $IJ = K$. According to Equations (5) and (10), we get

$$\begin{aligned} &\mathcal{K}^*(X, Y) \\ &= \mathcal{K}(X, Y) + 3|A_X Y|^2 \\ &= |X|^2|Y|^2 - \langle X, Y\rangle^2 \\ &\quad + 3\left(\langle XI, Y\rangle^2 + \langle XJ, Y\rangle^2 + \langle XK, Y\rangle^2\right). \end{aligned}$$

For the orthonormal vector fields X and Y, it follows from Equation (9), that

$$\mathcal{K}(X, Y) = 1 + 3\cos^2 \varphi(X, Y). \tag{11}$$

Thus, the sectional curvature $\mathcal{K}$ of the quaternion projective space $\mathbb{QP}^n$ satisfies $1 \leq \mathcal{K} \leq 4$. ∎

Definition 5.3: Consider a compact Riemannian manifold (M, g) with positive sectional curvature $\mathcal{K}$. The pinching constant is defined as follows:

$$\delta_m = \frac{\min \mathcal{K}(\sigma)}{\max \mathcal{K}(\sigma)},$$

where σ runs through all two-planes of T_pM and $p \in M$.

This means that the sectional curvature $\mathcal{K}$ obeys

$$\mathcal{K}_{\max} \geq \mathcal{K} \geq \delta_m \mathcal{K}_{\max} > 0.$$

Proposition 5.2 ([24]): *Let M be a compact, simply connected, Riemannian manifold with its sectional curvature $\mathcal{K}$ satisfying*

$$\mathcal{K}_{\max} \geq \mathcal{K} \geq \frac{1}{4}\mathcal{K}_{\max},$$

hence either M is homeomorphic to a sphere or isometric to one of the compact rank one symmetric spaces $\mathbb{C}P^n$, $\mathbb{Q}P^n$.

The pinching constant for the complex projective space $\mathbb{CP}^n$ and for the quaternion projective space $\mathbb{QP}^n$ is

$$\delta_{\mathbb{CP}^n} = \delta_{\mathbb{QP}^n} = \frac{1}{4}.$$

6. Conclusions

The projective spaces are considered in this article. Specifically, the real, complex and quaternion projective spaces are introduced. Some interesting observations and notions of these projective spaces are given. Indeed, we proved that their sectional curvatures are constant. The pinching constant for the complex projective space and for the quaternion projective space is determined.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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